Scalar and fuzzy cardinalities of crisp and fuzzy multisets*

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Abstract

In this paper we define in an axiomatic way scalar and fuzzy cardinalities of finite crisp and fuzzy multisets, and we obtain explicit descriptions for them.

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1 Introduction

A (crisp) *multiset* over a set of *types* V is simply a mapping $d : V \to \mathbb{N}$. A good survey of the mathematics of multisets, including their axiomatic foundation, can be found in [2]. Further results on the applications of multisets in several branches of computer science can be found in [7]. Multisets are also called *bags* in the literature [27].

The usual interpretation of a multiset $d : V \to \mathbb{N}$ is that it describes a set consisting of $d(v)$ "exact" copies of each type $v \in V$, without specifying which element of the set is a copy of which element of V; the number $d(v)$ is usually called the *multiplicity* of v in the multiset d. Notice in particular that the set described by the multiset does not contain any element that is not a copy of some $v \in V$, and that an element of it cannot be a copy of two different types.

A natural generalization of this interpretation of multisets leads to the notion of *fuzzy multiset*, or *fuzzy bag*, over a set of *types* V as a mapping $F : V \times [0, 1] \to \mathbb{N}$. Such a fuzzy multiset describes a set consisting of, for each $v \in V$ and for every $t \in [0, 1]$, $F(v, t)$ "possibly inexact" copies of v with degree of similarity t to it. In this paper we impose two restrictions on this interpretation of a fuzzy multiset, parallel to those highlighted in the crisp case, that allow us to slightly modify this definition. First, we assume that if an element of the set is an inexact copy of v with degree of similarity

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$t > 0$, then it cannot be an inexact copy of any other type in V with a non-negative degree of similarity. And second, the set described by the fuzzy multiset does not contain any element that is not a copy of some $v \in V$ with some non-negative degree of similarity. These two conditions entail that, for every $v \in V$, the value $F(v, 0)$ must be equal to $\sum_{w \in V-\{0\}} \sum_{t>0} F(w, t)$ and in particular that the restriction of F to $V \times \{0\}$ is determined by the restriction of F to $V \times]0, 1]$.

This restrictions allow us to define a *fuzzy multiset* over a set V as a mapping $F : V \times]0, 1] \to \mathbb{N}$, or, equivalently, as a mapping from V to the set $\mathbb{N}^{]0,1]}$ of all crisp multisets of $]0, 1]$, through the natural bijection $\mathbb{N}^{V \times]0,1]} \cong (\mathbb{N}^{]0,1]})^V$. Not having to care about the images under fuzzy multisets of the elements of the form $(v, 0)$ (which, anyway, are determined by the rest of images) will greatly simplify some of the definitions and results that will be introduced in the main body of this paper.

Our interest in fuzzy multisets stems from their application in the development of a fuzzy version of membrane computing that handles inexact copies of the objects used in computations. Membrane computing is a formal computational paradigm, invented in 1998 by G. Păun [17], that rewrites multisets of objects within a spatial structure inspired by the membrane structure of living cells. Despite its youth, membrane computing has become a very active branch of natural computing, as the textbook [18] and the 247 works cited in it witness.

In one of the simplest versions of membrane computing, and without entering into details, a *membrane system*, or also a *P-system*, consists of several *membranes* arranged in a hierarchical structure inside a main membrane and defining *regions* in-between them. These regions are assumed to contain multisets of objects called *reactives*. Computations on these multisets use *evolution rules* associated to each region, which may create, destroy or even move objects from one region to an adjacent one. These rules can be understood as formal models of biochemical processes involving chemical compounds represented by the reactives. At the end of a computation, the objects contained in the region defined by an *output membrane* are counted. The natural number obtained in this way is the result of this specific computation, and the set of all natural numbers obtained through all possible computations with a given membrane system is the set *generated* by this system. Full details of this and other versions of membrane computing can be found in [18].

In these "crisp" membrane systems, the contents of the regions defined by the membranes are multisets of reactives, and hence these regions are understood to contain only exact copies of the chemical compounds involved in the biochemical processes represented by the evolution rules. But, in a more realistic model, the objects contained in these regions would be inexact copies of the chemical compounds, and hence their contents would have to be described by means of fuzzy multisets over the set of reactives. Moreover, the evolution rules would produce also inexact copies of the resulting reactives, with a degree of similarity that would depend on the rule and the degree of similarity to the reactives of the objects used in the rule. We shall report in detail on the resulting fuzzy model of computation some time in the near future.

Anyway, as in the crisp case, at the end of a computation with such a fuzzy membrane system we will have to "count" the fuzzy multiset over the reactives contained in the output region. And this leads to the topic of this paper: how can we "count" fuzzy multisets?

The problem of "counting" fuzzy sets has generated a lot of literature since Zadeh's first definition of the cardinality of fuzzy sets [28]. In particular, the scalar cardinalities of fuzzy sets, which associate to each fuzzy set a positive real number, have been studied from the axiomatic point of view [9, 10, 12, 26] with the aim of capturing different ways of counting additive aspects of fuzzy sets like the cardinals of supports, of levels, of cores, etc. In a similar way, the fuzzy cardinalities of fuzzy sets [15, 20, 23, 24, 25], that associate to any fuzzy set a convex fuzzy natural number, have also been studied from the axiomatic point of view [8, 11].

As far as cardinalities of multisets goes, an extension to fuzzy multisets of Zadeh's original definition of the cardinality of fuzzy sets has already been introduced [1, 4, 27]. On the other hand, an extension to multisets of the concept of (convex) fuzzy cardinality for fuzzy sets has been used [5, 6] as well as nonconvex cardinalities of fuzzy multisets [14, 13].

In parallel with what has been done for fuzzy sets [26, 10], in this paper we introduce axiomatically two general ways of counting crisp and fuzzy finite multisets: the scalar cardinalities and fuzzy cardinalities. In both cases, and after their axiomatic introduction, we provide general explicit descriptions of them and we study their first properties. In particular, a (scalar or fuzzy) cardinality of a finite fuzzy multiset $F : V\times]0,1] \to \mathbb{N}$ will turn out to be always a sum of cardinalities of the crisp multisets $F(v,-) :]0,1] \to \mathbb{N}$.

2 Preliminaries

Let X be a crisp set. A (*crisp*) *multiset* over X is a mapping $M : X \to \mathbb{N}$, where $\mathbb{N}$ stands for the set of natural numbers including the 0. A multiset M over X is *finite* if its *support*

$$Supp(M) = \{x \in X | M(x) > 0\}$$

is a finite subset of X. We shall denote the sets of all multisets and of all finite multisets over a set X by $MS(X)$ and $FMS(X)$, respectively, and by $\bot$ the *null multiset*, defined by $\bot(x) = 0$ for every $x \in X$.

A *singleton* is a multiset over a set X that sends some element $x \in X$ to $1 \in \mathbb{N}$ and all other elements of X to $0 \in \mathbb{N}$; we shall denote such a singleton by $1/x$. More in general, we shall denote by n/x the multiset on X that sends $x \in X$ to $n \in \mathbb{N}$ and all other elements of X to 0: in particular, $0/x = \bot$ for every $x \in X$.

For every $A, B \in MS(X)$, their *sum* $A + B$ is the multiset

$$(A + B)(x) = A(x) + B(x), \quad x \in X.$$

Let us mention here that it has been argued [21, 22] that this sum $+$, also called *additive union*, is the right notion of union of multisets. Under the interpretation of multisets as sets of copies of types explained in the introduction, this sum corresponds to the disjoint union of sets, as it interprets that all copies of each x in the set represented by A are different from all copies of it in the set represented by B. This additive sum has properties quite different from the ordinary union of sets. For instance, the collection of submultisets of a given multiset is not closed under this operation and consequently no sensible notion of complement within this collection exists.

For every $A, B \in MS(X)$, their *join* $A \vee B$ and *meet* $A \wedge B$ are respectively the multisets over X defined pointwise by

$$(A \vee B)(x) = A(x) \vee B(x) \text{ and } (A \wedge B)(x) = A(x) \wedge B(x), \quad x \in X.$$

If A and B are finite, then $A + B$, $A \vee B$ and $A \wedge B$ are also finite.

The partial order $\leq$ on $MS(X)$ is defined by

$$A \leq B \text{ if and only if } A(x) \leq B(x) \text{ for every } x \in X.$$

If $A \leq B$, then their *difference* $B - A$ is the multiset defined pointwise by

$$(B - A)(x) = B(x) - A(x).$$

Let X be again a crisp set. A *fuzzy multiset* over X is a mapping $\overline{M} : X \to MS(]0, 1])$. A fuzzy multiset $\overline{M}$ over X is *finite* if its *support*

$$Supp(\overline{M}) = \{x \in X \mid \overline{M}(x) \neq \bot\}$$

is a finite subset of X and, for every $x \in Supp(\overline{M})$, $\overline{M}(x)$ is a finite multiset of $]0, 1]$. We shall denote the sets of all fuzzy multisets and of all finite fuzzy multisets over X by $\mathcal{FMS}(X)$ and $\mathcal{FFMS}(X)$, respectively, and by $\overline{\bot}$ the (finite) fuzzy multiset defined by $\overline{\bot}(x) = \bot$ for every $x \in X$.

For every $x \in X$ and $A \in MS(]0, 1])$, we shall denote by A/x the fuzzy multiset over X defined by $(A/x)(x) = A$ and $(A/x)(y) = \bot$ for every $y \neq x$. Notice that if A is finite, then A/x is also finite.

Given two fuzzy multisets $\overline{A}, \overline{B}$ over X, their *sum* $\overline{A} + \overline{B}$, their *join* $\overline{A} \vee \overline{B}$ and their *meet* $\overline{A} \wedge \overline{B}$ are respectively the fuzzy multisets over X defined pointwise by

$$\begin{aligned}(\overline{A} + \overline{B})(x) &= \overline{A}(x) + \overline{B}(x)\\ (\overline{A} \vee \overline{B})(x) &= \overline{A}(x) \vee \overline{B}(x)\\ (\overline{A} \wedge \overline{B})(x) &= \overline{A}(x) \wedge \overline{B}(x)\end{aligned}$$

where now the sum, join and meet on the right-hand side of these equalities are operations between multisets; so, for instance, $\overline{A}+\overline{B}: X \to MS(]0,1])$ is the fuzzy multiset such that

$$(\overline{A}+\overline{B})(x)(t) = \overline{A}(x)(t) + \overline{B}(x)(t) \quad \text{for every } x \in X \text{ and } t \in]0,1].$$

The partial order $\leq$ on $\mathcal{FMS}(X)$ is defined by

$$\overline{A} \leq \overline{B} \text{ if and only if } \overline{A}(x) \leq \overline{B}(x) \text{ for every } x \in X$$

where the symbol $\leq$ in the right-hand side of this equivalence stands for the partial order between crisp multisets defined above. If $\overline{A} \leq \overline{B}$, then their *difference* $\overline{B} - \overline{A}$ is the fuzzy multiset defined pointwise by

$$(\overline{B} - \overline{A})(x) = \overline{B}(x) - \overline{A}(x),$$

where, again, the difference in the right hand term in this equality stands for the difference of crisp multisets defined above.

A *generalized natural number* [25] is a fuzzy subset $\overline{n} : \mathbb{N} \to [0,1]$ of $\mathbb{N}$. Besides the usual union and intersection of fuzzy subsets, we shall use the following operation $\oplus$ on $[0,1]^{\mathbb{N}}$, called the *extended sum* (see for instance [24]): for every generalized natural numbers $\overline{m}, \overline{n}$,

$$(\overline{n} \oplus \overline{m})(k) = \bigvee \{\overline{n}(i) \wedge \overline{m}(j) \mid i+j=k\} \text{ for every } k \in \mathbb{N}.$$

It is well known that this extended sum of generalized natural numbers is associative, commutative and that if $\overline{0}$ denotes the generalized natural number that sends 0 to 1 and every $n > 0$ to 0, then $\overline{n} \oplus \overline{0} = \overline{n}$ for every generalized natural number $\overline{n}$. As a consequence of these properties, the extended sum of m generalized natural numbers is well defined:

$$(\overline{n}_1 \oplus \cdots \oplus \overline{n}_m)(i) = \bigvee \{\overline{n}_1(i_1) \wedge \cdots \wedge \overline{n}_m(i_m) \mid i_1 + i_2 + \cdots + i_m = i\}.$$

A generalized natural number is *convex* when $\overline{n}(k) \geq \overline{n}(i) \wedge \overline{n}(j)$ for every $i \leq k \leq j$. We shall denote by $\overline{\mathbb{N}}$ the set of all convex generalized natural numbers. Every increasing or decreasing generalized natural number is convex, and the extended sum of two convex generalized natural number is again convex. Moreover, the extended sum of two increasing (resp., decreasing) generalized natural numbers is again increasing (resp., decreasing). For these and other properties of generalized natural numbers, see [24].

3 Scalar cardinalities of finite multisets over $]0,1]$

We introduce and discuss in this section the notion of scalar cardinality of crisp finite multisets on $]0,1]$. From now on, $\mathbb{R}^+$ stands for the set of all real numbers greater or equal than 0.

Definition 1 *A* scalar cardinality *on* $FMS(]0,1])$ *is a mapping* $Sc : FMS(]0,1]) \to \mathbb{R}^+$ *that satisfies the following conditions:*

(i) $Sc(A+B) = Sc(A) + Sc(B)$ *for every* $A, B \in FMS(]0,1])$.

(ii) $Sc(1/1) = 1$.

Remark 1 *If* $Sc : FMS(]0,1]) \to \mathbb{R}^+$ *is a scalar cardinality, then* $Sc(\perp) = 0$*, because*

$$1 = Sc(1/1) = Sc((1/1) + \perp) = Sc(1/1) + Sc(\perp) = 1 + Sc(\perp),$$

and if $A \le B$*, then* $Sc(A) \le Sc(B)$*, because in this case*

$$Sc(B) = Sc(A + (B - A)) = Sc(A) + Sc(B - A) \ge Sc(A).$$

Remark 2 *We have that if* Sc *is a scalar cardinality on* $FMS(]0,1)]$*, then, for every* $A, B \in FMS(]0,1])$*,*

$$Sc(A \vee B) + Sc(A \wedge B) = Sc(A) + Sc(B),$$

because

$$A \vee B + A \wedge B = A + B$$

and then the additivity of scalar cardinalities (condition (i) in Definition 1) applies. In particular, if $A \wedge B = \perp$*, then* $Sc(A \vee B) = Sc(A) + Sc(B)$.

Next proposition provides a description of all scalar cardinalities on $FMS(]0,1)$.

Proposition 1 *A mapping* $Sc : FMS(]0,1]) \to \mathbb{R}^+$ *is a scalar cardinality if and only if there exists some mapping* $f :]0,1] \to \mathbb{R}^+$ *with* $f(1) = 1$*, such that*

$$Sc(A) = \sum_{t \in Supp(A)} f(t)A(t) \qquad \textit{for every } A \in FMS(]0,1]).$$

Proof. Let Sc be a scalar cardinality on $FMS(]0,1])$, and consider the mapping

$$\begin{array}{rrcl} f: &]0,1] & \to & \mathbb{R}^+ \\ & t & \mapsto & Sc(1/t) \end{array}$$

We have that $f(1) = Sc(1/1) = 1$, by condition (ii) in Definition 1. And since every $A \in FMS(]0,1])$ can be decomposed into a sum of singletons, namely,

$$A = \sum_{t \in Supp(A)} \overbrace{1/t + \cdots + 1/t}^{A(t)},$$

condition (i) in Definition 1 implies that

$$Sc(A) = \sum_{t \in Supp(A)} \overbrace{Sc(1/t) + \cdots + Sc(1/t)}^{A(t)} = \sum_{t \in Supp(A)} A(t) f(t).$$

Conversely, let $f :]0,1] \to \mathbb{R}^+$ be a mapping such that $f(1) = 1$, and let $Sc_f : FMS(]0,1]) \to \mathbb{R}^+$ be the mapping defined by

$$Sc_f(A) = \sum_{t \in Supp(A)} f(t) A(t)$$

for every $A \in FMS(]0,1])$. Then, this mapping satisfies the defining conditions of scalar cardinalities. Indeed, $Sc(1/1) = f(1) = 1$, which proves condition (ii) in Definition 1. As far as condition (i) goes,

$$\begin{aligned} Sc(A+B) &= \textstyle\sum_{t \in Supp(A+B)} f(t)(A(t) + B(t)) \\ &= \textstyle\sum_{t \in Supp(A+B)} f(t)A(t) + \sum_{t \in Supp(A+B)} f(t)B(t) \\ &= \textstyle\sum_{t \in Supp(A)} f(t)A(t) + \sum_{t \in Supp(B)} f(t)B(t) = Sc_f(A) + Sc_f(B). \end{aligned}$$

■

Henceforth, and as we did in the last proof, whenever we want to stress the mapping $f :]0,1] \to \mathbb{R}^+$ that *generates* a given scalar cardinality, we shall denote the latter by Sc_f. In particular, Sc_1 will denote from now on the scalar cardinality associated to the constant mapping 1, i.e.,

$$Sc_1(A) = \sum_{t \in Supp(A)} A(t) \quad \text{for every } A \in FMS(]0,1]).$$

Let Sc_f be any scalar cardinality on $FMS(]0,1])$. As we saw in Remark 1, for every $A, B \in FMS(]0,1])$, if $A \leq B$, then $Sc_f(A) \leq Sc_f(B)$. The converse implication is, of course, false. Let, for instance, $f :]0,1] \to \mathbb{R}^+$ be the constant mapping 1, and let A be the singleton $1/t_0$ and B the singleton $1/t_1$ with $t_0 \neq t_1$. Then $Sc_f(A) = 1 = Sc_f(B)$ but neither $A \leq B$ nor $B \leq A$.

It is more interesting to point out that, for certain mappings f, it may happen that $A \leq B$ and $Sc_f(A) = Sc_f(B)$ but $A \neq B$. For instance, let $f :]0,1] \to \mathbb{R}^+$ be any mapping such that $f(t_0) = 0$ and $f(1) = 1$ for some $t_0 \neq 1$. Let A be the singleton $1/t_0$ and B the multiset $2/t_0$. Then $A \leq B$ and $Sc_f(A) = 0 = Sc_f(B)$, but $A \neq B$.

Actually, sending some element of $]0,1]$ to 0 is unavoidable in order to obtain such a counterexample: the reader may easily prove that if $f :]0,1] \to \mathbb{R}^+$ is such that $f(t) > 0$ for every $t \in]0,1]$, then, for every $A, B \in FMS(]0,1])$, if $A \leq B$ and $Sc_f(A) = Sc_f(B)$, then $A = B$.

4 Fuzzy cardinalities of finite multisets over $]0,1]$

In this section, we introduce and discuss the notion of fuzzy cardinality of crisp finite multisets over $]0,1]$.

Definition 2 *A* fuzzy cardinality *on $FMS(]0,1])$ is a mapping $\mathcal{C} : FMS(]0,1] \to \overline{\mathbb{N}}$ that satisfies the following conditions:*

(i) (Additivity) For every $A, B \in FMS(]0,1])$, $\mathcal{C}(A+B) = \mathcal{C}(A) \oplus \mathcal{C}(B)$.

(ii) (Variability) For every $A, B \in FMS(]0,1])$ and for every $i > Sc_1(A)$ and $j > Sc_1(B)$, $\mathcal{C}(A)(i) = \mathcal{C}(B)(j)$.

(iii) (Consistency) If $Supp(A) \subseteq \{1\}$, then $\mathcal{C}(A)(i) \in \{0,1\}$ for every $i \in \mathbb{N}$ and, moreover, if $n = A(1)$, then $\mathcal{C}(A)(n) = 1$.

(iv) (Monotonicity) If $t, t' \in]0,1]$ are such that $t \leq t'$, then

$$\mathcal{C}(1/t)(0) \geq \mathcal{C}(1/t')(0) \quad \text{and} \quad \mathcal{C}(\perp)(1) \leq \mathcal{C}(1/t)(1) \leq \mathcal{C}(1/t')(1).$$

Let us explain the meaning as well as some motivations for each one of these conditions. The additivity property claims for an "extension principle" version for convex generalized natural number of the additivity of the classical cardinal of a crisp multiset and thus it seems quite natural. With respect to variability, it is a consequence of the idea that the elements t not belonging to the support of a finite multiset A should not affect the cardinality of A. In particular, it is required that the value of the cardinality of any finite fuzzy multiset A must be the same for all natural numbers greater than $\sum_{t \in]0,1]} A(t)$ and for every A. Consistency requires that, on each multiset of the form $n/1$, with $n \in \mathbb{N}$, any fuzzy cardinality must take values only in $\{0,1\}$, and the value 1 on the specific number n. Finally, monotonicity captures the restriction that the value of the cardinality of singletons on 0 must decrease and their value on 1 must increase with the element of their support.

The fuzzy cardinality defined in the next example will play a key role henceforth, and, as we shall see in Section 5, it generalizes in a very precise way the usual bracket notation for fuzzy sets.

Example 2 *Let us consider the function*

$$\begin{array}{rrcl} [\,]: & FMS(]0,1]) & \to & [0,1]^{\mathbb{N}} \\ & A & \mapsto & [A] \end{array}$$

where, for every $A \in FMS(]0,1])$,

$$\begin{array}{rrcl} [A]: & \mathbb{N} & \to & [0,1] \\ & i & \mapsto & [A]_i \end{array}$$

is defined by

$$[A]_i = \bigvee\{t \in [0,1] \mid \sum_{t' \geq t} A(t') \geq i\}.$$

It is clear that $[A]$ *is decreasing for every* $A \in FMS(]0,1])$*: for every* $i \leq j$*,*

$$\{t \in [0,1] \mid \sum_{t' \geq t} A(t') \geq j\} \subseteq \{t \in [0,1] \mid \sum_{t' \geq t} A(t') \geq i\}$$

and hence

$$[A]_j = \bigvee\{t \in [0,1] \mid \sum_{t' \geq t} A(t') \geq j\} \leq \bigvee\{t \in [0,1] \mid \sum_{t' \geq t} A(t') \geq i\} = [A]_i.$$

Therefore, $[A] \in \overline{\mathbb{N}}$ *for every* $A \in FMS(]0,1])$*.*

The mapping $[\ ]$ *satisfies the variability (if* $i > Sc_1(A)$*, then* $[A]_i = \bigvee \emptyset = 0$*), the consistency (for every* $n \in \mathbb{N}$*,* $[n/1]_i$ *is* 1 *for every* $i \leq n$ *and* 0 *for every* $i > n$*) and the monotonicity (for every* $t \in]0,1]$*,* $[1/t]_0 = 1$ *and* $[1/t]_1 = t$*) conditions. As far as the additivity condition goes, we have that, for every* $A, B \in FMS(]0,1])$ *and for every* $i \in \mathbb{N}$*,*

$$\begin{aligned}
[A+B]_i &= \bigvee\{t \in]0,1] \mid \textstyle\sum_{t' \geq t}(A+B)(t') \geq i\} \\
&= \bigvee\{t \in]0,1] \mid \textstyle\sum_{t' \geq t} A(t') + \sum_{t' \geq t} B(t') \geq i\} \\
&= \bigvee\{t \in]0,1] \mid \textit{there exist } j,k \in \mathbb{N} \textit{ such that } j+k=i \\
&\qquad\qquad \textit{and } \textstyle\sum_{t' \geq t} A(t') \geq j \textit{ and } \sum_{t' \geq t} B(t') \geq k\} \\
&= \bigvee\Big\{\bigvee\{t \in]0,1] \mid \textstyle\sum_{t' \geq t} A(t') \geq j \textit{ and } \sum_{t' \geq t} B(t') \geq k\} \\
&\qquad\qquad \Big| \; j,k \in \mathbb{N}, j+k=i\Big\} \\
&= \bigvee\Big\{\bigvee\{t \in]0,1] \mid \textstyle\sum_{t' \geq t} A(t') \geq j\} \wedge \bigvee\{t \in]0,1] \mid \sum_{t' \geq t} B(t') \geq k\} \\
&\qquad\qquad \Big| \; j,k \in \mathbb{N},\ j+k=i\Big\} \\
&= \bigvee\{[A]_j \wedge [B]_k \mid j,k \in \mathbb{N},\ j+k=i\} = ([A] \oplus [B])(i).
\end{aligned}$$

Therefore, $[\ ]$ *is a fuzzy cardinality on* $FMS(]0,1])$*.*

In Theorem 7 below we shall need a detailed description of $[\ ]$*, which we provide now. If* $A = \perp$*, then* $[A]_0 = 1$ *and* $[A]_i = 0$ *for every* $i \geq 1$*. Let now* A *be a non-null finite multiset over* $]0,1]$*, say with* $Supp(A) = \{t_1, \ldots, t_n\} \neq \emptyset$*,* $t_1 < \cdots < t_n$*. In this case it* *is straightforward to check that*

$$\sum_{t'\geq t} A(t') = \begin{cases} \sum_{j=1}^{n} A(t_j) & \text{if } t \in [0, t_1] \\ \sum_{j=2}^{n} A(t_j) & \text{if } t \in]t_1, t_2] \\ \vdots & \\ \sum_{j=s}^{n} A(t_j) & \text{if } t \in]t_{s-1}, t_s] \\ \vdots & \\ A(t_n) & \text{if } t \in]t_{n-1}, t_n] \\ 0 & \text{if } t \in]t_n, 1] \end{cases}$$

Therefore, for every $i \geq 0$,

$$[A]_i = \begin{cases} 1 & \text{if } i = 0 \\ t_n & \text{if } 0 < i \leq A(t_n) \\ t_{n-1} & \text{if } A(t_n) < i \leq A(t_n) + A(t_{n-1}) \\ \vdots & \\ t_s & \text{if } \sum_{j=s+1}^{n} A(t_j) < i \leq \sum_{j=s}^{n} A(t_j) \\ \vdots & \\ t_1 & \text{if } \sum_{j=2}^{n} A(t_j) < i \leq \sum_{j=1}^{n} A(t_j) \\ 0 & \text{if } \sum_{j=1}^{n} A(t_j) < i \end{cases}$$

The following technical lemma will be used henceforth several times.

Lemma 3 *Let $\mathcal{C} : FMS(]0,1]) \to \overline{\mathbb{N}}$ be a fuzzy cardinality and let A be a non-null multiset over $]0,1]$. Then, for every $k \in \mathbb{N}$,*

$$\mathcal{C}(A)(k) = \bigvee \Big\{ \bigwedge_{t\in Supp(A)} \mathcal{C}(1/t)(i_{t,1}) \wedge \cdots \wedge \mathcal{C}(1/t)(i_{t,A(t)}) \mid \sum_{t\in Supp(A)} \sum_{l=1}^{A(t)} i_{t,l} = k \Big\}.$$

Proof. This is a direct consequence of the additivity of $\mathcal{C}$ and the fact that A decomposes into

$$A = \sum_{t\in Supp(A)} \overbrace{1/t + \cdots + 1/t}^{A(t)}.$$

■

Corollary 4 *For every $n \in \mathbb{N}$ and for every $k < n$, $\mathcal{C}(n/1)(k) = \mathcal{C}(1/1)(0)$.*

Proof. By the previous lemma, we have that

$$\mathcal{C}(n/1)(k) = \bigvee \Big\{ \mathcal{C}(1/1)(i_1) \wedge \cdots \wedge \mathcal{C}(1/1)(i_n) \mid i_1 + \cdots + i_n = k \Big\}. \qquad (1)$$

Since the only decomposition of 0 as a sum of natural numbers is as a sum of 0's, this equality implies that if $n > 0$, then

$$\mathcal{C}(n/1)(0) = \overbrace{\mathcal{C}(1/1)(0) \wedge \cdots \wedge \mathcal{C}(1/1)(0)}^{n} = \mathcal{C}(1/1)(0).$$

On the other hand, the consistency condition implies that $\mathcal{C}(1/1)(1) = 1$, and the variability condition that $\mathcal{C}(1/1)(j)$ is either 0 or 1 for every $j \geq 2$. In the first case, it is clear that, for every $k = 1, \ldots, n-1$, all terms of the form

$$\mathcal{C}(1/1)(i_1) \wedge \cdots \wedge \mathcal{C}(1/1)(i_n)$$

with $i_1 + \cdots + i_n = k$ are 0 except

$$\overbrace{\mathcal{C}(1/1)(1) \wedge \cdots \wedge \mathcal{C}(1/1)(1)}^{k} \wedge \overbrace{\mathcal{C}(1/1)(0) \wedge \cdots \wedge \mathcal{C}(1/1)(0)}^{n-k} = 1 \wedge \mathcal{C}(1/1)(0) = \mathcal{C}(1/1)(0),$$

and hence, by (1), $\mathcal{C}(n/1)(k) = \mathcal{C}(1/1)(0)$. In the second case, i.e., if $\mathcal{C}(1/1)(j) = 1$ for every $j \geq 2$, every term of the form $\mathcal{C}(1/1)(i_1) \wedge \cdots \wedge \mathcal{C}(1/1)(i_n)$ with $i_1 + \cdots + i_n = k$ is the meet of some 1's and at least one $\mathcal{C}(1/1)(0)$ and hence it is equal to $\mathcal{C}(1/1)(0)$. This, again by (1), entails that $\mathcal{C}(n/1)(k) = \mathcal{C}(1/1)(0)$. ∎

Our next goal is to obtain explicit description of all fuzzy cardinalities on $FMS(]0,1])$. We shall actually provide two such explicit descriptions, in Theorems 6 and 7. To begin with, next proposition introduces a family of fuzzy cardinalities that, as we shall see, will cover all possible fuzzy cardinalities.

Proposition 5 *Let $f : [0,1] \to [0,1]$ be an increasing mapping such that $f(0) \in \{0,1\}$ and $f(1) = 1$ and let $g : [0,1] \to [0,1]$ be a decreasing mapping such that $g(0) = 1$ and $g(1) \in \{0,1\}$. Let $\mathcal{C}_{f,g} : FMS(]0,1]) \to [0,1]^{\mathbb{N}}$ be the mapping defined on $\perp$ by $\mathcal{C}_{f,g}(\perp)(0) = 1$ and $\mathcal{C}_{f,g}(\perp)(i) = f(0)$ for every $i \geq 1$; on every singleton $1/t$, $t \in]0,1]$, by*

$$\mathcal{C}_{f,g}(1/t)(0) = g(t),\ \mathcal{C}_{f,g}(1/t)(1) = f(t),\ \mathcal{C}_{f,g}(1/t)(i) = f(0) \text{ for every } i \geq 2;$$

and on every $A \in FMS(]0,1]) - \{\perp\}$ by means of

$$\mathcal{C}_{f,g}(A) = \bigoplus_{t \in Supp(A)} \overbrace{\mathcal{C}_{f,g}(1/t) \oplus \cdots \oplus \mathcal{C}_{f,g}(1/t)}^{A(t)}.$$

Then, $\mathcal{C}_{f,g}$ is a fuzzy cardinality on $FMS(]0,1])$, which will be called from now on the fuzzy cardinality generated by f and g.

Proof. Let f and g be a pair of functions satisfying the conditions in the statement. To prove that $\mathcal{C}_{f,g}(A) \in \overline{\mathbb{N}}$ for every $A \in FMS(]0,1])$, we must prove that it is convex, and since $\mathcal{C}(\perp)$ is clearly convex (it is decreasing) and the extended sum of convex generalized natural numbers is convex, it is enough to prove the convexity of every $\mathcal{C}_{f,g}(1/t)$ with $t \in]0,1]$. Finally, since each $\mathcal{C}_{f,g}(1/t)$ is constant on $\{n \in \mathbb{N} \mid n \geq 2\}$, to prove that it is convex it is enough to check the convexity condition on $\{0,1,2\}$. And, indeed,

$$\mathcal{C}_{f,g}(1/t)(1) = f(t) \geq f(0) \geq g(t) \wedge f(0) = \mathcal{C}_{f,g}(1/t)(0) \wedge \mathcal{C}_{f,g}(1/t)(2),$$

because f is increasing.

Therefore, $\mathcal{C}_{f,g}$ takes values in $\overline{\mathbb{N}}$. Now, we must check that it satisfies conditions (i) to (iv) in Definition 2. To simplify the notations, once fixed the mappings f and g, we shall denote $\mathcal{C}_{f,g}$ by $\mathcal{C}$ throughout the rest of the proof.

(i) Let $A, B \in FMS(]0,1])$.

Assume first that one of them, say B, is the null multiset $\perp$. We must prove that $\mathcal{C}(A) = \mathcal{C}(A) \oplus \mathcal{C}(\perp)$. Now we distinguish two cases.

If $f(0) = 0$, then

$$(\mathcal{C}(A) \oplus \mathcal{C}(\perp))(k) = \bigvee\{\mathcal{C}(A)(i) \wedge \mathcal{C}(\perp)(k-i) \mid i = 0,\dots,k\},$$

and since $\mathcal{C}(\perp)(0) = 1$ and $\mathcal{C}(\perp)(j) = 0$ for every $j \geq 1$, we have that $(\mathcal{C}(A) \oplus \mathcal{C}(\perp))(k) = \mathcal{C}(A)(0) \wedge \mathcal{C}(\perp)(0) = \mathcal{C}(A)(0)$ and, for every $k \geq 1$,

$$(\mathcal{C}(A) \oplus \mathcal{C}(\perp))(k) = \bigvee\{\mathcal{C}(A)(0) \wedge 0, \dots, \mathcal{C}(A)(k-1) \wedge 0, \mathcal{C}(A)(k) \wedge 1\} = \mathcal{C}(A)(k).$$

If $f(0) = 1$, then, since f is increasing, $f(t) = 1$ for every $t \in [0,1]$, and therefore $\mathcal{C}(1/t)$ is increasing for every $t \in]0,1]$. Since the extended sum of increasing generalized natural numbers is increasing, the definition of $\mathcal{C}(A)$ for every $A \neq \perp$ implies that it is increasing. Moreover, $\mathcal{C}(\perp)$ is in this case the constant mapping 1, which is increasing. Therefore, in this case $\mathcal{C}(A)$ is increasing for every $A \in FMS(]0,1])$. Now,

$$\begin{aligned}(\mathcal{C}(A) \oplus \mathcal{C}(\perp))(k) &= \bigvee\{\mathcal{C}(A)(i) \wedge \mathcal{C}(\perp)(k-i) \mid i = 0,\dots,k\}\\ &= \bigvee\{\mathcal{C}(A)(i) \wedge 1 \mid i = 0,\dots,k\}\\ &= \bigvee\{\mathcal{C}(A)(0),\dots,\mathcal{C}(A)(k)\} = \mathcal{C}(A)(k),\end{aligned}$$

because $\mathcal{C}(A)$ is increasing.

This proves that $\mathcal{C}(A) = \mathcal{C}(A) \oplus \mathcal{C}(\perp)$ for every $A \in FMS(]0,1])$. Assume now that A and B are both non-null. Then, the associativity and the commutativity of the extended sum in $\overline{\mathbb{N}}$ imply that

$$\begin{aligned}
\mathcal{C}(A+B) &= \bigoplus_{t\in Supp(A+B)} \overbrace{\mathcal{C}(1/t)\oplus\cdots\oplus\mathcal{C}(1/t)}^{A(t)+B(t)}\\
&= \Big(\bigoplus_{t\in Supp(A+B)} \overbrace{\mathcal{C}(1/t)\oplus\cdots\oplus\mathcal{C}(1/t)}^{A(t)}\Big)\oplus\Big(\bigoplus_{t\in Supp(A+B)} \overbrace{\mathcal{C}(1/t)\oplus\cdots\oplus\mathcal{C}(1/t)}^{B(t)}\Big)\\
&=\Big(\bigoplus_{t\in Supp(A)} \overbrace{\mathcal{C}(1/t)\oplus\cdots\oplus\mathcal{C}(1/t)}^{A(t)}\Big)\oplus\Big(\bigoplus_{t\in Supp(B)} \overbrace{\mathcal{C}(1/t)\oplus\cdots\oplus\mathcal{C}(1/t)}^{B(t)}\Big)\\
&= \mathcal{C}(A)\oplus\mathcal{C}(B)
\end{aligned}$$

(ii) We will prove that if $i > Sc_1(A)$, then $\mathcal{C}(A)(i) = f(0)$, which belongs to $\{0,1\}$. If $A = \perp$ it is given by the very definition of $\mathcal{C}(\perp)$, so assume that $A \neq \perp$. In this case, if $i > Sc_1(A)$, then

$$\begin{aligned}
\mathcal{C}(A)(i) &= \Big(\bigoplus_{t\in Supp(A)} \overbrace{\mathcal{C}(1/t)\oplus\cdots\oplus\mathcal{C}(1/t)}^{A(t)}\Big)(i)\\
&= \bigvee\Big\{\bigwedge_{t\in Supp(A)} \mathcal{C}(1/t)(i_{t,1})\wedge\cdots\wedge\mathcal{C}(1/t)(i_{t,A(t)})\Big|\sum_{t\in Supp(A)}\sum_{j=1}^{A(t)} i_{t,j} = i\Big\}
\end{aligned}$$

Now, notice that if $i > Sc_1(A)$, then in every decomposition of i as the sum of $Sc_1(A)$ natural numbers there must be at least one summand greater or equal than 2: this means that in each expression

$$\bigwedge_{t\in Supp(A)} \mathcal{C}(1/t)(i_{t,1})\wedge\cdots\wedge\mathcal{C}(1/t)(i_{t,A(t)}) \text{ with } \sum_{t\in Supp(A)}\sum_{j=1}^{A(t)} i_{t,j} = i$$

there is at least one $\mathcal{C}(1/t)(i_{t,i})$ equal to $f(0)$.

If $f(0) = 0$, this implies that each such expression is 0 and hence its join is still $0 = f(0)$. On the other hand, if $f(0) = 1$ then, since $1 = f(0) \leq f(t) \leq 1$ for every $t \in [0,1]$ (because f is increasing and it takes values in $[0,1]$), we have that $f(t) = 1$ and hence $\mathcal{C}(1/t)(1) = 1$ for every $t \in [0,1]$. Now, take a decomposition of i as $\sum_{t\in Supp(A)}\sum_{j=1}^{A(t)} i_{t,j}$ with all $i_{t,j} \geq 1$, which will always exist, and it will happen that

$$\bigwedge_{t\in Supp(A)} \mathcal{C}(1/t)(i_{t,1})\wedge\cdots\wedge\mathcal{C}(1/t)(i_{t,A(t)}) = 1,$$

and therefore the join of all these expressions will be also $1 = f(0)$.

(iii) If $A = n/1$ with $n > 0$, then

$$\mathcal{C}(A) = \overbrace{\mathcal{C}(1/1) \oplus \cdots \oplus \mathcal{C}(1/1)}^{n},$$

and hence

$$\mathcal{C}(A)(i) = \bigvee \{\mathcal{C}(1/1)(i_1) \wedge \cdots \wedge \mathcal{C}(1/1)(i_n) \mid i_1 + \cdots + i_n = i\}.$$

Since $\mathcal{C}(1/1)(0) = g(1) \in \{0,1\}$, $\mathcal{C}(1/1)(1) = f(1) = 1$ and $\mathcal{C}(1/1)(j) = f(0) \in \{0,1\}$ for every $j \geq 2$, it is clear that $\mathcal{C}(A)(i) \in \{0,1\}$ for every $i \geq 0$. As far as the specific value of $\mathcal{C}(A)(n)$ goes, notice that n can de decomposed as the sum of n 1's. Therefore, the set of real numbers whose join yields $\mathcal{C}(A)(n)$ contains

$$\overbrace{\mathcal{C}(1/1)(1) \wedge \cdots \wedge \mathcal{C}(1/1)(1)}^{n} = \mathcal{C}(1/1)(1) = f(1) = 1,$$

and hence this maximum is 1.

As far as the case when $A = 0/1 = \perp$ goes, we have that $\mathcal{C}(A)(0) = 1$ and $\mathcal{C}(A)(i) \in \{0,1\}$ for every $i \geq 1$ by the very definition of $\mathcal{C}(\perp)$

(iv) It holds by the definition of $\mathcal{C}(1/t)$, $t \in]0,1]$, and $\mathcal{C}(\perp)$ and the properties of f and g.

■

Next theorem is one of our main results and it shows that every fuzzy cardinality on $FMS(]0,1])$ belongs to the family of fuzzy cardinalities described in the previous proposition.

Theorem 6 *A mapping $\mathcal{C} : FMS(]0,1]) \to I^{\mathbb{N}}$ is a fuzzy cardinality if and only if $\mathcal{C} = \mathcal{C}_{f,g}$ for some increasing mapping $f : [0,1] \to [0,1]$ such that $f(0) \in \{0,1\}$ and $f(1) = 1$ and some decreasing mapping $g : [0,1] \to [0,1]$ such that $g(0) = 1$ and $g(1) \in \{0,1\}$.*

Proof. The "if" implication, i.e., that every mapping of the form $\mathcal{C}_{f,g}$ for f and g as in the statement is a fuzzy cardinality, is proved in Proposition 5.

Conversely, let $\mathcal{C} : FMS(]0,1]) \to \overline{\mathbb{N}}$ be a fuzzy cardinality. Consider the mappings $f, g : [0,1] \to [0,1]$ defined, for every $t \in]0,1]$, by

$$f(t) = \mathcal{C}(1/t)(1), \quad g(t) = \mathcal{C}(1/t)(0),$$

and let $f(0) = \mathcal{C}(\perp)(1)$ and $g(0) = 1$.

Let us prove that these functions satisfy the properties required in the statement.

- f is increasing by condition (iv) in Definition 2.

- g is decreasing on $]0,1]$ by the same condition (iv) in Definition 2, and since $g(0)=1$, it is clear that it is decreasing on the whole interval $[0,1]$.

- By condition (iii) in that definition, we have that $g(1)=\mathcal{C}(1/1)(0)\in\{0,1\}$, $f(1)=\mathcal{C}(1/1)(1)=1$ and $f(0)=\mathcal{C}(0/1)(1)\in\{0,1\}$.

Finally, let us prove that $\mathcal{C}=\mathcal{C}_{f,g}$. It is clear that $\mathcal{C}(\bot)=\mathcal{C}_{f,g}(\bot)$. Moreover, $\mathcal{C}(1/t)=\mathcal{C}_{f,g}(1/t)$ for every $t\in]0,1]$, because

$$\mathcal{C}(1/t)(0)=g(t)=\mathcal{C}_{f,g}(1/t)(0),\ \mathcal{C}(1/t)(1)=f(t)=\mathcal{C}_{f,g}(1/t)(1),$$
$$\mathcal{C}(1/t)(i)=\mathcal{C}(\bot)(1)=f(0)=\mathcal{C}_{f,g}(1/t)(i) \text{ for every } i\geq 2$$

(the equality $\mathcal{C}(1/t)(i)=\mathcal{C}(\bot)(1)$ is a consequence of condition (ii) in Definition 2). And then, the additivity of fuzzy cardinalities (condition (i) in Definition 2) entails that, for every $A\in FMS(]0,1])-\{\bot\}$,

$$\begin{aligned}\mathcal{C}(A) &= \bigoplus_{t\in Supp(A)} \overbrace{\mathcal{C}(1/t)\oplus\cdots\oplus\mathcal{C}(1/t)}^{A(t)}\\ &= \bigoplus_{t\in Supp(A)} \overbrace{\mathcal{C}_{f,g}(1/t)\oplus\cdots\oplus\mathcal{C}_{f,g}(1/t)}^{A(t)}=\mathcal{C}_{f,g}(A).\end{aligned}$$

■

Now, we give an explicit description of all fuzzy cardinalities in terms of the fuzzy cardinality [ ].

Theorem 7 *Let $f:[0,1]\to[0,1]$ be an increasing mapping such that $f(0)\in\{0,1\}$ and $f(1)=1$, let $g:[0,1]\to[0,1]$ be a decreasing mapping such that $g(0)=1$ and $g(1)\in\{0,1\}$, and let $\mathcal{C}$ be the fuzzy cardinality $\mathcal{C}_{f,g}$ on $FMS(]0,1])$ generated by these mappings. Then,*

$$\mathcal{C}(A)(i)=f([A]_i)\wedge g([A]_{i+1})$$

for every $A\in FMS(]0,1])$ and $i\in\mathbb{N}$.

Proof. We shall distinguish the cases $f(0)=1$ and $f(0)=0$. Although it is not formally necessary, the case when $f(0)=1$ is simpler and its discussion may enlighten the computations in the other case.

1) To begin with, assume that $f(0)=1$. Then, f being increasing, f is the constant mapping 1 and then $\mathcal{C}=\mathcal{C}_{1,g}$. We want to prove in this case that

$$\mathcal{C}(A)(i)=g([A]_{i+1})$$

for every $A\in FMS(]0,1])$ and $i\in\mathbb{N}$.

For $\perp$ we have that $\mathcal{C}(\perp)(i) = 1 = g(0) = g([\perp]_{i+1})$ for every $i \geq 0$. Let us prove now the desired equality for non-null multisets.

Let A be a multiset with $Supp(A) = \{t_1, \ldots, t_n\} \neq \emptyset$, $t_1 < \cdots < t_n$. From the explicit description of [ ] given in Example 2, we have that

$$g([A]_{i+1}) = \begin{cases} g(t_n) & \text{if } 0 < i+1 \leq A(t_n), \text{ i.e., if } 0 \leq i < A(t_n) \\ g(t_{n-1}) & \text{if } A(t_n) < i+1 \leq A(t_n) + A(t_{n-1}), \text{ i.e.,} \\ & \quad \text{if } A(t_n) \leq i < A(t_n) + A(t_{n-1}) \\ \vdots & \\ g(t_s) & \text{if } \sum_{j=s+1}^n A(t_j) < i+1 \leq \sum_{j=s}^n A(t_j), \text{ i.e.,} \\ & \quad \text{if } \sum_{j=s+1}^n A(t_j) \leq i < \sum_{j=s}^n A(t_j) \\ \vdots & \\ g(t_1) & \text{if } \sum_{j=2}^n A(t_j) < i+1 \leq \sum_{j=1}^n A(t_j), i.e., \\ & \quad \text{if } \sum_{j=2}^n A(t_j) \leq i < \sum_{j=1}^n A(t_j) \\ 1 & \text{if } \sum_{j=1}^n A(t_j) < i+1, \text{ i.e., if } \sum_{j=1}^n A(t_j) \leq i \end{cases}$$

On the other hand, by Lemma 3 we have that, for every $i \geq 0$,

$$\mathcal{C}(A)(i) = \bigvee \Big\{ \bigwedge_{j=1}^n \mathcal{C}(1/t_j)(i_{j,1}) \wedge \cdots \wedge \mathcal{C}(1/t_j)(i_{j,A(t_j)}) \mid \sum_{j=1}^n \sum_{l=1}^{A(t_j)} i_{j,l} = i \Big\}$$

In every expression

$$\bigwedge_{j=1}^n \mathcal{C}(1/t_j)(i_{j,1}) \wedge \cdots \wedge \mathcal{C}(1/t_j)(i_{j,A(t_j)}) \tag{2}$$

every $\mathcal{C}(1/t_j)(i_{j,l})$ with $i_{j,l} \geq 1$ is 1: if $i_{j,l} = 1$, it is $f(t_j) = 1$ and if $i_{j,l} \geq 2$, it is $f(0) = 1$. Therefore, when we compute the meet in (2), all these 1's disappear and this expression is either equal to 1 (if every $i_{j,l} > 0$ in it) or to some

$$\mathcal{C}(1/t_{j_1})(0) \wedge \cdots \wedge \mathcal{C}(1/t_{j_k})(0) = g(t_{j_1}) \wedge \cdots \wedge g(t_{j_k}) = g(t_{j_k})$$

for some $j_1, \ldots, j_k \in \{1, \ldots, n\}$ such that $t_{j_1} < \cdots < t_{j_k}$ (these are exactly the indexes j such that $i_{j,l} = 0$ for some l); in the last equality we have used that g is decreasing.

We use this remark to prove that $\mathcal{C}(A)(i) = g([A]_{i+1})$ on every interval which we have split $\mathbb{N}$ into in the explicit description of $g([A]_{i+1})$ given above.

- If $i \geq \sum_{j=1}^n A(t_j)$, then there exists a decomposition of i as a sum $i_{1,1}+\cdots+i_{n,A(t_n)}$ with $i_{j,l} > 0$ for every $j = 1, \ldots, n$ and $l = 1, \ldots, A(t_j)$, which entails that $\mathcal{C}(A)(i) = 1$.

- If $\sum_{j=2}^{n} A(t_j) \leq i < \sum_{j=1}^{n} A(t_j)$, then there exists a decomposition of i as a sum $i_{1,1} + \cdots + i_{n,A(t_n)}$ with $i_{j,l} = 1$ for every $j > 1$ and for every $l = 1, \ldots, A(t_j)$, and some $i_{1,l} = 0$. The expression (2) corresponding to this decomposition is equal to $g(t_1)$, and for any other decomposition of i this expression is equal to some $g(t_j)$ with $j \geq 1$ (because every decomposition of i uses some 0). Since g is decreasing, the join of all these terms, and hence $\mathcal{C}(A)(i)$, is $g(t_1)$.

- If $\sum_{j=3}^{n} A(t_j) \leq i < \sum_{j=2}^{n} A(t_j)$, then there exists a decomposition of i as a sum $i_{1,1} + \cdots + i_{n,A(t_n)}$ with $i_{j,l} = 1$ for every $j > 2$ and for every $l = 1, \ldots, A(t_j)$, and some $i_{2,l} = 0$. The expression (2) corresponding to this decomposition is equal to $g(t_2)$, and this expression is equal to some $g(t_j)$ with $j \geq 2$ for any other decomposition of i (because there cannot exist any decomposition of i with less or equal than $A(t_1)$ 0's). Since g is decreasing, the join of all these terms, and hence $\mathcal{C}(A)(i)$, is $g(t_2)$.

- In general, for every $s = 1, \ldots, n-1$, if

$$\sum_{j=s+1}^{n} A(t_j) \leq i < \sum_{j=s}^{n} A(t_j),$$

  there exists a decomposition of i as a sum $i_{1,1} + \cdots + i_{n,A(t_n)}$ with $i_{j,l} = 1$ for every $j > s$ and for every $l = 1, \ldots, A(t_j)$ and some $i_{s,l} = 0$. The expression (2) corresponding to this decomposition is equal to $g(t_s)$, and this expression is equal to some $g(t_j)$ with $j \geq s$ for any other decomposition of i (because there cannot exist any decomposition of i with less or equal than $A(t_1) + \cdots + A(t_{s-1})$ 0's). Since g is decreasing, the join of all these terms is $g(t_s)$, and hence $\mathcal{C}(A)(i) = g(t_s)$.

- Finally, if

$$0 \leq i < A(t_n),$$

  every decomposition of i as a sum $i_{1,1} + \cdots + i_{n,A(t_n)}$ must have some $i_{n,l} = 0$. Therefore, every expression (2) in this case is equal to $g(t_n)$ and hence $\mathcal{C}(A)(i) = g(t_n)$.

This finishes the proof in the case $f(0) = 1$.

2) Let us assume now that $f(0) = 0$. For $\perp$ we have that $\mathcal{C}(\perp)(0) = 1 = f([\perp]_0) \wedge g([\perp]_1)$ and $\mathcal{C}(\perp)(i) = 0 = f([\perp]_i) \wedge g([\perp]_{i+1})$ for every $i \geq 1$. Let us prove now the equality in the statement for non-null multisets.

Let A be a multiset with $Supp(A)=\{t_1,\dots,t_n\}\neq\emptyset$, $t_1<\cdots<t_n$. From the explicit description of [ ] given in Example 2, we have that

$$f([A]_i)\wedge g([A]_{i+1})=\begin{cases} g(t_n) & \text{if } i=0\\ f(t_n)\wedge g(t_n) & \text{if } 0<i<A(t_n)\\ f(t_n)\wedge g(t_{n-1}) & \text{if } i=A(t_n)\\ f(t_{n-1})\wedge g(t_{n-1}) & \text{if } 0<i<A(t_n)\\ \vdots & \\ f(t_{s+1})\wedge g(t_s) & \text{if } i=\sum_{j=s+1}^n A(t_j)\\ f(t_s)\wedge g(t_s) & \text{if } \sum_{j=s+1}^n A(t_j)<i<\sum_{j=s}^n A(t_j)\\ \vdots & \\ f(t_1)\wedge g(t_1) & \text{if } \sum_{j=2}^n A(t_j)<i<\sum_{j=1}^n A(t_j)\\ f(t_1) & \text{if } i=\sum_{j=1}^n A(t_j)\\ 0 & \text{if } \sum_{j=1}^n A(t_j)<i \end{cases}$$

On the other hand, by Lemma 3,

$$\mathcal{C}(A)(i)=\bigvee\Big\{\bigwedge_{j=1}^n \mathcal{C}(1/t_j)(i_{j,1})\wedge\cdots\wedge\mathcal{C}(1/t_j)(i_{j,A(t_j)})\mid \sum_{j=1}^n\sum_{l=1}^{A(t_j)} i_{j,l}=i\Big\}$$

for every $i\geq 0$. Recall that $\mathcal{C}(1/t_j)(0)=g(t_j)$, $\mathcal{C}(1/t_j)(1)=f(t_j)$ and $\mathcal{C}(1/t_j)=0$ for every $i\geq 2$ and then, in particular

$$\bigwedge_{j=1}^n \mathcal{C}(1/t_j)(i_{j,1})\wedge\cdots\wedge\mathcal{C}(1/t_j)(i_{j,A(t_j)}) \tag{3}$$

is 0 whenever some $i_{j,l}$ is greater or equal than 2. On the other hand, for every decomposition of i as a sum $\sum_{j=1}^n\sum_{l=1}^{A(t_j)} i_{j,l}$ with all $i_{j,l}\leq 1$, expression (3) will be equal to

$$f(t_{j_1})\wedge g(t_{j_2}),$$

where j_1 is the lowest index j such that some $i_{j,l}$ is 1, and j_2 is the highest index j such that some $i_{j,l}$ is 0 (if every $i_{j,l}$ is 1, then it will be $f(t_1)$, and if every $i_{j,l}$ is 0, then it will be $g(t_n)$).

Let us check now that $\mathcal{C}(A)(i)=f([A]_i)\wedge g([A]_{i+1})$ on each interval which we have divided $\mathbb{N}$ into in the explicit description of the values $f([A]_i)\wedge g([A]_{i+1})$ given above.

- If $\sum_{j=1}^n A(t_j)<i$, then every decomposition of i as a sum $\sum_{j=1}^n\sum_{l=1}^{A(t_j)} i_{j,l}$ involves some $i_{j,l}\geq 2$. As we have just pointed out, this implies that $\mathcal{C}(A)(i)=0$.

- If $i = \sum_{j=1}^{n} A(t_j)$, then the only decomposition of i as a sum $\sum_{j=1}^{n} \sum_{l=1}^{A(t_j)} i_{j,l}$ that does not involve any $i_{j,l} \geq 2$ is the one with all summands 1. For this decomposition, as we have just mentioned, the expression (3) will be equal to $f(t_1)$, and this will be the maximum $\mathcal{C}(A)(i)$ of all such expressions for this value of i.

- In general, if $\sum_{j=s+1}^{n} A(t_j) < i < \sum_{j=s}^{n} A(t_j)$ for some $s = 1, \ldots, n$, then there exists a decomposition of i as a sum $\sum_{j=1}^{n} \sum_{l=1}^{A(t_j)} i_{j,l}$ such that $i_{j,l} = 0$ for every $j < s$, and there are l_1, l_2 such that $i_{s,l_1} = 1$ and $i_{s,l_2} = 0$, and $i_{j,l} = 1$ for every $j > s$. For this decomposition, the expression (3) is equal to $f(t_s) \wedge g(t_s)$.

  And any decomposition without this form will have either some $i_{j,l} = 0$ with $j > s$ or some $i_{k,l} = 1$ with $k < s$, and it will give (3) a value of the form $f(t_k) \wedge g(t_j)$ with $k < s$ and $j \geq s$ or with $k \leq s$ and $j > s$. Since f is increasing and g is decreasing, it is clear that $f(t_s) \wedge g(t_s)$ will be the maximum of all these possible values, and hence $\mathcal{C}(A)(i)$.

- In general, if $i = \sum_{j=s}^{n} A(t_j)$ for some $s = 1, \ldots, n$, then we can decompose i as $\sum_{j=1}^{n} \sum_{l=1}^{A(t_j)} i_{j,l}$ with $i_{j,l} = 1$ for every $j \geq s$ and $i_{j,l} = 0$ for every $j < s$. For this decomposition, the expression (3) is equal to $f(t_s) \wedge g(t_{s-1})$.

  And any other decomposition of i will have some $i_{j,l} = 0$ with $j \geq s$ and some $i_{k,l} = 1$ with $k < s$, and it will give (3) a value of the form $f(t_k) \wedge g(t_j)$ with $k < s$ and $j \geq s$. Since f is increasing and g is decreasing, it is clear that $f(t_s) \wedge g(t_{s-1})$ will be the maximum of all these possible values, and hence $\mathcal{C}(A)(i)$.

- If $0 < i < A(t_n)$, then there exists a decomposition of i as a sum $\sum_{j=1}^{n} \sum_{l=1}^{A(t_j)} i_{j,l}$ such that $i_{j,l} = 0$ for every $j < n$ and there are l_1, l_2 such that $i_{n,l_1} = 1$ and $i_{n,l_2} = 0$. For this decomposition, the expression (3) is equal to $f(t_n) \wedge g(t_n)$. And any other decomposition will have some $i_{n,l_2} = 0$, and hence it will give (3) a value $f(t_j) \wedge g(t_n)$. Since f is increasing, it is clear that $f(t_n) \wedge g(t_n)$ will be the maximum of all these values, and hence $\mathcal{C}(A)(i)$.

- Finally, if $i = 0$, then the only decomposition of i as a sum $\sum_{j=1}^{n} \sum_{l=1}^{A(t_j)} i_{j,l}$ is the one with all summands 0. For this decomposition, as we have just mentioned, the expression (3) will be equal to $g(t_n)$, and this will be clearly $\mathcal{C}(A)(i)$.

This shows that $\mathcal{C}(A)(i) = f([A]_i) \wedge g([A]_{i+1})$ for every $i \geq 0$. ■

We shall now study in detail the increasing and decreasing fuzzy cardinalities. Last theorem will entail that any other fuzzy cardinality will be built up from cardinalities of these two types in a simple way.

Definition 3 *A fuzzy cardinality* $\mathcal{C} : FMS(]0,1]) \to \overline{\mathbb{N}}$ *is* increasing *(resp.,* decreasing*) if and only if* $\mathcal{C}(A) \in \overline{\mathbb{N}}$ *is an increasing (resp., decreasing) mapping for every* $A \in FMS(]0,1])$.

Proposition 8 *Let* $\mathcal{C} : FMS(]0,1]) \to \overline{\mathbb{N}}$ *be the fuzzy cardinality* $\mathcal{C}_{f,g}$. *Then, the following assertions are equivalent:*

(i) $\mathcal{C}$ *is increasing.*

(ii) $g(t) \le f(t) \le f(0)$ *for every* $t \in]0,1]$.

(iii) f *is the constant mapping* 1.

(iv) $\mathcal{C}(A)(k) = g([A]_{k+1})$ *for every* $A \in FMS(]0,1])$ *and* $k \in \mathbb{N}$.

Proof. (i)$\Longrightarrow$(ii) Let $\mathcal{C} = \mathcal{C}_{f,g}$ be an increasing fuzzy cardinality. Then, for every $t \in]0,1]$, $\mathcal{C}(1/t)$ is an increasing generalized natural number, and in particular

$$g(t) = \mathcal{C}(1/t)(0) \le f(t) = \mathcal{C}(1/t)(1) \le f(0) = \mathcal{C}(1/t)(2).$$

(ii)$\Longrightarrow$(iii) The assumption that $f(1) = 1$ implies, by (ii), that $f(0) = 1$ and then, since f is increasing on $[0,1]$, $f(t) = 1$ for every $t \in [0,1]$.

(iii)$\Longrightarrow$(iv) By Theorem 7, if f is the constant mapping 1, then

$$\mathcal{C}(A)(k) = 1 \wedge g([A]_{k+1}) = g([A]_{k+1}) \text{ for every } A \in FMS(]0,1]) \text{ and } k \in \mathbb{N}.$$

(iv)$\Longrightarrow$(i) The fuzzy cardinality [ ] is decreasing (see Example 2) and g is also a decreasing mapping, and hence it is clear from (iv) that every $\mathcal{C}(A)$ is increasing. ■

Proposition 9 *Let* $\mathcal{C} : FMS(]0,1]) \to \overline{\mathbb{N}}$ *be the fuzzy cardinality* $\mathcal{C}_{f,g}$. *Then, the following assertions are equivalent:*

i) $\mathcal{C}$ *is a decreasing cardinality.*

ii) $g(t) \ge f(t)$ *for every* $t \in]0,1]$.

iii) $g(t)$ *is the constant mapping 1.*

(iv) $\mathcal{C}(A)(k) = f([A]_k)$ *for every* $A \in FMS(]0,1])$ *and* $k \in \mathbb{N}$.

Proof. (i)$\Longrightarrow$(ii) Let $\mathcal{C} = \mathcal{C}_{f,g}$ be a decreasing fuzzy cardinality. Then, for every $t \in]0,1]$, $\mathcal{C}(1/t)$ is an decreasing generalized natural number, and in particular

$$g(t) = \mathcal{C}(1/t)(0) \ge f(t) = \mathcal{C}(1/t)(1).$$

(ii)$\Longrightarrow$(iii) The assumption that $f(1) = 1$ implies, by (ii), that $g(1) = 1$ and then, since $g(0) = 1$ and g is decreasing, it must happen that $g(t) = 1$ for every $t \in [0, 1]$.

(iii)$\Longrightarrow$(iv) By Theorem 7, if g is the constant mapping 1, then

$$\mathcal{C}(A)(k) = f([A]_k) \wedge 1 = f([A]_k) \text{ for every } A \in FMS(]0, 1]) \text{ and } k \in \mathbb{N}.$$

(iv)$\Longrightarrow$(i) Since the fuzzy cardinality [ ] is decreasing and f is increasing,it is clear from the description of $\mathcal{C}(A)$ given in (iv) that it is decreasing. ∎

Remark 3 *Notice that the only fuzzy cardinality which is both decreasing and increasing is $\mathcal{C}_{1,1}$, which is given by $\mathcal{C}_{1,1}(A)(k) = 1$ for every $A \in FMS(]0, 1])$ and $k \in \mathbb{N}$.*

Example 10 *If we take g to be the constant mapping 1 and f the identity on $[0, 1]$, then $\mathcal{C}_{f,g}$ is the fuzzy cardinality defined by*

$$\mathcal{C}_{f,g}(A)(i) = [A]_i \textit{ for every } A \in FMS(]0, 1]) \textit{ and } i \in \mathbb{N};$$

i.e., $\mathcal{C}_{\mathrm{Id},1}$ is the fuzzy cardinality [ ] in Example 2.

Example 11 *If we take g to be the constant mapping 1 and $f_a : [0, 1] \to [0, 1]$, with $a \in [0, 1]$, the mapping defined by $f_a(t) = 0$ for every $t < a$ and $f_a(t) = 1$ for every $t \geq a$, then*

$$\mathcal{C}_{f_a,g}(A)(i) = f_a([A]_i) = \begin{cases} 0 & \textit{if } [A]_i < a \\ 1 & \textit{if } [A]_i \geq a \end{cases}$$

for every $A \in FMS(]0, 1])$ and $i \in \mathbb{N}$; i.e., $\mathcal{C}_{f,g}(A)(i)$ is 0 if $\sum_{t' \geq t} A(t') \geq i$ implies $t < a$, and it is 1 if there exists some $t \geq a$ such that $\sum_{t' \geq t} A(t') \geq i$ (recall from Example 2 that the mapping $t \mapsto \sum_{t' \geq t} A(t')$ is constant on a finite number of intervals of the form $]t_i, t_{i+1}]$ that cover $]0, 1]$).

Example 12 *If we take f to be the constant mapping 1 and $g : [0, 1] \to [0, 1]$ the mapping defined by $g(t) = 1 - t$, then $\mathcal{C}_{f,g}(A)(i) = 1 - [A]_{i+1}$ for every $A \in FMS(]0, 1])$ and every $i \in \mathbb{N}$.*

Example 13 *If we take f to be the constant mapping 1 and $g_a : [0, 1] \to [0, 1]$, with $a \in [0, 1]$, the mapping defined by $g_a(t) = 1$ for every $t < a$ and $g_a(t) = 0$ for every $t \geq a$, then*

$$\mathcal{C}_{f,g_a}(A)(i) = g_a([A]_{i+1}) = \begin{cases} 1 & \textit{if } [A]_{i+1} < a \\ 0 & \textit{if } [A]_{i+1} \geq a \end{cases}$$

for every $A \in FMS(]0, 1])$ and for every $i \in \mathbb{N}$.

Corollary 14 *Every fuzzy cardinality on $FMS(]0, 1])$ is the meet of an increasing fuzzy cardinality and a decreasing fuzzy cardinality.*

Proof. Theorem 7 can be rewritten as $\mathcal{C}_{f,g} = \mathcal{C}_{f,1} \wedge \mathcal{C}_{1,g}$, where $\mathcal{C}_{1,g}$ is increasing by Proposition 8 and $\mathcal{C}_{f,1}$ is decreasing by Proposition 9. ∎

Remark 4 *We have proved that, for every $A \in FMS(]0,1])$ and $k \in \mathbb{N}$,*

$$\begin{aligned} \mathcal{C}_{f,g}(A)(k) &= \mathcal{C}_{f,1}([A])(k) \wedge \mathcal{C}_{1,g}([A])(k+1) \\ &= \begin{cases} \mathcal{C}_{f,1}([A])(k) & \text{if } \mathcal{C}_{f,1}([A])(k) \leq \mathcal{C}_{1,g}([A])(k+1) \\ \mathcal{C}_{1,g}([A])(k+1) & \text{if } \mathcal{C}_{1,g}([A])(k+1) \leq \mathcal{C}_{f,1}([A])(k) \end{cases} \end{aligned}$$

Since $\mathcal{C}_{f,1}$ is decreasing and $\mathcal{C}_{1,g}$ is increasing, we have that if $\mathcal{C}_{1,g}([A])(k+1) \leq \mathcal{C}_{f,1}([A])(k)$ for some k, then $\mathcal{C}_{1,g}([A])(i+1) \leq \mathcal{C}_{f,1}([A])(i)$ for every $i \leq k$, and that if $\mathcal{C}_{f,1}([A])(k) \leq \mathcal{C}_{1,g}([A])(k+1)$ for some k, then $\mathcal{C}_{f,1}([A])(i) \leq \mathcal{C}_{1,g}([A])(i+1)$ for every $i \geq k$. This implies that, in the non-trivial cases neither f nor g are the constant mapping 1, there exists an $n_0 \in \mathbb{N}$ such that $\mathcal{C}_{f,g}(A)$ is given by (the increasing mapping) $\mathcal{C}_{1,g}(A)$ on $\{i \in \mathbb{N} \mid i < n_0\}$ and by (the decreasing mapping) $\mathcal{C}_{f,1}(A)$ on $\{i \in \mathbb{N} \mid i \geq n_0\}$.

Corollary 15 *If $\mathcal{C} : FMS(]0,1]) \to \overline{\mathbb{N}}$ is a fuzzy cardinality and $A, B \in FMS(]0,1])$, then*

$$\mathcal{C}(A \vee B) \oplus \mathcal{C}(A \wedge B) = \mathcal{C}(A) \oplus \mathcal{C}(B).$$

In particular, if $A \wedge B = \perp$, then $\mathcal{C}(A \vee B) = \mathcal{C}(A) \oplus \mathcal{C}(B)$.

Proof. It is obvious that, for every $A, B \in FMS(]0,1])$,

$$A \wedge B + A \vee B = A + B.$$

The first assertion in the statement is a direct consequence then of this equality and the additivity of fuzzy cardinalities.

Now, if $A \wedge B = \perp$, we have that $\mathcal{C}(A \vee B) \oplus \mathcal{C}(\perp) = \mathcal{C}(A) \oplus \mathcal{C}(B)$, and in the proof of the additivity of $\mathcal{C}_{f,g}$ in Proposition 5 we have proved that $\mathcal{C}_{f,g}(A) \oplus \mathcal{C}_{f,g}(\perp) = \mathcal{C}_{f,g}(A)$ for every fuzzy cardinality $\mathcal{C}_{f,g}$ and every $A \in FMS(]0,1])$. Since every fuzzy cardinality $\mathcal{C}$ has the form $\mathcal{C}_{f,g}$, this entails that $\mathcal{C}(A \vee B) \oplus \mathcal{C}(\perp) = \mathcal{C}(A \vee B)$ and hence that $\mathcal{C}(A \vee B) = \mathcal{C}(A) \oplus \mathcal{C}(B)$, as we claimed. ■

Corollary 16 *The meet of two fuzzy cardinalities on $FMS(]0,1])$ is again a fuzzy cardinality.*

Proof. Let $\mathcal{C}_{f,g}$ and $\mathcal{C}_{f',g'}$ be the fuzzy cardinalities associated to the mappings $f, g : [0,1] \to [0,1]$ and $f', g' : [0,1] \to [0,1]$, respectively. We have just proved that $\mathcal{C}_{f,g} = \mathcal{C}_{f,1} \wedge \mathcal{C}_{1,g}$ and $\mathcal{C}_{f',g'} = \mathcal{C}_{f',1} \wedge \mathcal{C}_{1,g'}$, and hence, by the associativity of the meet operation $\wedge$ in $\overline{\mathbb{N}}$,

$$\mathcal{C}_{f,g} \wedge \mathcal{C}_{f',g'} = (\mathcal{C}_{f,1} \wedge \mathcal{C}_{1,g}) \wedge (\mathcal{C}_{f',1} \wedge \mathcal{C}_{1,g'}) = (\mathcal{C}_{f,1} \wedge \mathcal{C}_{f',1}) \wedge (\mathcal{C}_{1,g} \wedge \mathcal{C}_{1,g'}). \quad (4)$$

Now, if $f, f' : [0,1] \to [0,1]$ are two increasing mappings such that $f(0), f'(0) \in \{0,1\}$ and $f(1) = f'(1) = 1$, then their meet

$$\begin{array}{rccl} f \wedge f' : & [0,1] & \mapsto & [0,1] \\ & t & \mapsto & f(t) \wedge f'(t) \end{array}$$

is also an increasing mapping that sends 0 to either 0 or 1, and 1 to 1. And it is clear from Theorem 7 that $\mathcal{C}_{f,1} \wedge \mathcal{C}_{f',1} = \mathcal{C}_{f\wedge f',1}$.

In a similar way, if $g, g' : [0,1] \to [0,1]$ are two decreasing mappings such that $g(0) = g'(0) = 1$ and $g(1), g'(1) \in \{0,1\}$, then

$$\begin{array}{rccl} g \wedge g' : & [0,1] & \mapsto & [0,1] \\ & t & \mapsto & g(t) \wedge g'(t) \end{array}$$

is also a decreasing mapping such that $(g \wedge g')(0) = 1$ and $(g \wedge g')(1) \in \{0,1\}$ and, again by Theorem 7, $\mathcal{C}_{1,g} \wedge \mathcal{C}_{1,g'} = \mathcal{C}_{1,g\wedge g',1}$.

Therefore, from (4) and these observations we deduce that

$$\mathcal{C}_{f,g} \wedge \mathcal{C}_{f',g'} = \mathcal{C}_{f\wedge f',1} \wedge \mathcal{C}_{1,g\wedge g'} = \mathcal{C}_{f\wedge f',g\wedge g'}$$

is a fuzzy cardinality. ∎

Remark 5 *It is interesting to point out that the join of two fuzzy cardinalities need not be a fuzzy cardinality; actually, it need not even take values in* $\overline{\mathbb{N}}$. *For instance, consider* $\mathcal{C} = \mathcal{C}_{\mathrm{Id},1} \vee \mathcal{C}_{1,1-\mathrm{Id}}$, *where* $\mathrm{Id} : [0,1] \to [0,1]$ *stands for the identity. Thus,* $\mathcal{C} : FMS(]0,1]) \to \overline{\mathbb{N}}$ *is defined by*

$$\mathcal{C}(A)(i) = [A]_i \wedge (1 - [A]_{i+1}) \quad \text{for every } A \in FMS(]0,1]) \text{ and } i \in \mathbb{N}.$$

Now, let A *be* $1/t_1 + 1/t_2$ *with* $t_1 < t_2 < 1$. *Then, by Example 2,*

$$\mathcal{C}(A)(0) = 1, \quad \mathcal{C}(A)(1) = t_2 \wedge (1 - t_1) \neq 1, \quad \mathcal{C}(A)(i) = 1,$$

which is not convex.

We define now a partial order $\preceq$ on fuzzy cardinalities that is reminiscent of the partial order on $\overline{\mathbb{N}}$ defined in [24].

Definition 4 *Let* $f, g : [0,1] \to [0,1]$ *be mappings as in Proposition 5 and let* $\mathcal{C}_{f,g}$ *the fuzzy cardinality generated by them. The partial order* $\preceq$ *on* $\mathcal{C}_{f,g}(FMS(]0,1])) \subseteq \overline{\mathbb{N}}$ *is defined as follows: for every* $A, B \in FMS(]0,1])$, $\mathcal{C}_{f,g}(A) \preceq \mathcal{C}_{f,g}(B)$ *if and only if* $f([A]_i) \leq f([B]_i)$ *and* $g([A]_{i+1}) \geq g([B]_{i+1})$ *for every* $i \in \mathbb{N}$.

It is straightforward to check that $\preceq$ is a partial order.

Remark 6 *Let $\leq$ be the usual partial order on $[0,1]^{\mathbb{N}}$, defined by*

$$F \leq G \text{ if and only if } F(n) \leq G(n) \text{ for every } n \in \mathbb{N}.$$

If $f:[0,1] \to [0,1]$ is an increasing mapping as in Proposition 5, then $\mathcal{C}_{f,1}(A) \preceq \mathcal{C}_{f,1}(B)$ if and only if $\mathcal{C}_{f,1}(A) \leq \mathcal{C}_{f,1}(B)$. But if $g:[0,1] \to [0,1]$ is an decreasing mapping as in Proposition 5, then $\mathcal{C}_{1,g}(A) \preceq \mathcal{C}_{1,g}(B)$ if and only if $\mathcal{C}_{1,g}(A) \geq \mathcal{C}_{1,g}(B)$. And, in general, the description of $\mathcal{C}_{f,g}$ given in Remark 4 shows that if $\mathcal{C}_{f,g}(A) \preceq \mathcal{C}_{f,g}(B)$, then $\mathcal{C}_{f,g}(B)(k) \leq \mathcal{C}_{f,g}(A)(k)$ for every k in a certain initial interval of $\mathbb{N}$ and, after that, $\mathcal{C}_{f,g}(A)(k) \leq \mathcal{C}_{f,g}(B)(k)$. Thus, there is no relation between $\preceq$ and $\leq$.

Proposition 17 *If $A, B \in FMS(]0,1])$ are such that $[A]_i \leq [B]_i$ for every $i \in \mathbb{N}$, then $\mathcal{C}_{f,g}(A) \preceq \mathcal{C}_{f,g}(B)$ for every $f, g : [0,1] \to [0,1]$. In particular, if $A \leq B$ as multisets, then $\mathcal{C}_{f,g}(A) \preceq \mathcal{C}_{f,g}(B)$ for every $f, g : [0,1] \to [0,1]$.*

Proof. Since f is increasing and g is decreasing, $[A]_i \leq [B]_i$ for every $i \in \mathbb{N}$, implies that $f([A]_i) \leq f([B]_i)$ and $g([A]_{i+1}) \geq g([B]_{i+1})$ for every $i \in \mathbb{N}$. ∎

As it is usual, it is straightforward to produce examples showing that, in general, $\mathcal{C}_{f,g}(A) \preceq \mathcal{C}_{f,g}(B)$ does not imply $A \leq B$.

Let us end this section with two last properties of fuzzy cardinalities.

Proposition 18 *Let $\mathcal{C}$ be a fuzzy cardinality on $FMS(]0,1])$. If $A, B \in MS(]0,1])$ are such that $A \leq B$, then the equation*

$$\mathcal{C}(A) \oplus \alpha = \mathcal{C}(B)$$

has a solution in $\overline{\mathbb{N}}$, and one such solution is $\mathcal{C}(A - B)$.

Proof. Since $A + (B - A) = B$, the additivity of fuzzy cardinalities entails that $\mathcal{C}(A) \oplus \mathcal{C}(B - A) = \mathcal{C}(B)$. ∎

Proposition 19 *Let $f : [0,1] \to [0,1]$ be an injective and increasing mapping such that $f(0) = 0$ and $f(1) = 1$, and let $g : [0,1] \to [0,1]$ be an injective and decreasing mapping such that $g(0) = 1$ and $g(1) = 0$, Then, for every $A, B \in FMS(]0,1])$, $[A]_i = [B]_i$ for every $i \in \mathbb{N}$ if and only if $\mathcal{C}_{f,g}(A) = \mathcal{C}_{f,g}(B)$.*

Proof. The "only if" implication is a direct consequence of Theorem 7; actually, if $[A]_i = [B]_i$ for every $i \in \mathbb{N}$, then $\mathcal{C}(A) = \mathcal{C}(B)$ *for every* fuzzy cardinality $\mathcal{C}$.

As far as the "if" implication goes, by Remark 4 for every $A \in FMS(]0,1])$ there exists some $n_A \in \mathbb{N}$ such that if $i < n_A$ then such that $\mathcal{C}(A) = g([A]_{i+1})$ and if $i \geq n_A$, then $\mathcal{C}(A) = f([A]_i)$. If $\mathcal{C}_{f,g}(A) = \mathcal{C}_{f,g}(B)$, then $n_A = n_B$ and this implies that $g([A]_{i+1}) = g([B]_{i+1})$ for every $i \geq n_A$ and $f([A]_i) = f([B]_i)$ for every $i > n_A$ and hence, since f and g are injective, $[A]_i = [B]_i$ for every $i \geq 1$. Since, $[A]_0 = 1 = [B]_0$, the equality $[A]_i = [B]_i$ holds for every $i \in \mathbb{N}$. ∎

5 Multisets defined by fuzzy sets

Let $F :\to [0,1]$ a fuzzy set that is *finite*, in the sense that its *support*

$$Supp(F) = \{x \in X \mid F(t) \neq 0\}$$

is finite. This fuzzy set *defines* a finite multiset

$$\begin{array}{rccl} M_F : &]0,1] & \to & \mathbb{N} \\ & t & \mapsto & |F^{-1}(t)| \end{array}$$

where $|\ |$ denotes the usual cardinality of a crisp set. Notice that if X is arbitrary, then $|F^{-1}(0)|$ can be infinite, and hence M_F cannot be defined in general on 0.

As a fuzzy set, F has scalar and fuzzy cardinalities. Namely, for every increasing mapping $f : [0,1] \to [0,1]$ such that $f(0) = 0$ and $f(1) = 1$, we have the scalar cardinality [26] $\widehat{Sc_f}(F) \in \mathbb{R}^+$ defined by

$$\widehat{Sc_f}(F) = \sum_{x \in Supp(X)} f(F(x))$$

and, for every increasing mapping $f : [0,1] \to [0,1]$ such that $f(0) \in \{0,1\}$ and $f(1) = 1$ and for every decreasing mapping $g : [0,1] \to [0,1]$ such that $g(0) = 1$ and $g(1) \in \{0,1\}$, we have [11] the fuzzy cardinality $\widehat{\mathcal{C}_{f,g}}(F) \in \overline{\mathbb{N}}$ defined by

$$\widehat{\mathcal{C}_{f,g}}(F)(i) = f([F]_i) \wedge g([F]_{i+1}) \text{ for every } i \in \mathbb{N},$$

where now $[F]_i$ stands for

$$[F]_i = \bigvee \{t \in [0,1] \mid |\{x \in X \mid F(x) \geq t\}| \geq i\}.$$

One immediately notes that for every $f : [0,1] \to [0,1]$ for which we define a scalar cardinality $\widehat{Sc_f}$ on fuzzy sets on X, we have defined a scalar cardinality Sc_f on multisets over $]0,1]$, and that for every $f, g : [0,1] \to [0,1]$ for which we define a fuzzy cardinality $\widehat{\mathcal{C}_{f,g}}$ on fuzzy sets on X, we have also defined a fuzzy cardinality $\mathcal{C}_{f,g}$ on multisets over $]0,1]$. One can ask then whether there is a relation between a cardinality of F and the corresponding cardinality of M_F. Next propositions answer this question.

Proposition 20 *Let $f : [0,1] \to [0,1]$ be an increasing mapping such that $f(0) = 0$ and $f(1) = 1$. Let $\widehat{Sc_f}$ the scalar cardinality on fuzzy sets of X generated by f and let Sc_f be the scalar cardinality on multisets of $]0,1]$ generated by f. Then, for every fuzzy set F on X,*

$$\widehat{Sc_f}(F) = Sc_f(M_F).$$

Proof. A simple computation shows that

$$\begin{aligned} Sc_f(M_F) &= \sum_{t\in Supp(M_F)} f(t)M_F(t) = \sum_{t\in Supp(M_F)} f(t)|F^{-1}(t)| \\ &= \sum_{t\in F(X)} \overbrace{f(t)+\cdots+f(t)}^{|F^{-1}(t)|} = \sum_{x\in Supp(F)} f(F(x)) = \widehat{Sc_f}(F). \end{aligned}$$

■

Proposition 21 *Let $f:[0,1]\to[0,1]$ be an increasing mapping such that $f(0)\in\{0,1\}$ and $f(1)=1$ and let $g:[0,1]\to[0,1]$ be a decreasing mapping such that $g(0)=1$ and $g(1)\in\{0,1\}$. Let $\widehat{\mathcal{C}_{f,g}}$ be the fuzzy cardinality on fuzzy sets of X generated by f and g and let $\mathcal{C}_{f,g}$ be the fuzzy cardinality on multisets of $]0,1]$ generated by f and g. Then, for every fuzzy set F on X,*

$$\widehat{\mathcal{C}_{f,g}}(F) = \mathcal{C}_{f,g}(M_F).$$

Proof. By definition,

$$\mathcal{C}_{f,g}(M_F)(i) = f([M_F]_i \wedge g([M_F]_{i+1}),$$

where

$$\begin{aligned} [M_F]_i &= \bigvee\{t\in[0,1] \mid \sum_{t'\geq t} M_F(t') \geq i\} \\ &= \bigvee\{t\in[0,1] \mid \sum_{t'\geq t} |F^{-1}(t')| \geq i\} \\ &= \bigvee\{t\in[0,1] \mid |\{x\in X \mid F(x)\geq t\}| \geq i\} = [F]_i \end{aligned}$$

This shows that

$$\mathcal{C}_{f,g}(M_F)(i) = f([F]_i \wedge g([F]_{i+1}) = \widehat{\mathcal{C}_{f,g}}(F)(i).$$

■

6 Scalar and fuzzy cardinalities of finite fuzzy multisets

Let us recall that a finite fuzzy multiset over a set X is a mapping $\overline{M}: X \to FMS(]0,1])$ such that

$$Supp(\overline{M}) = \{x\in X \mid \overline{M}(x) \neq \bot\}$$

is finite. The set of all finite fuzzy multisets over X is denoted by $\mathcal{FFMS}(X)$. For every $x\in X$ and for every $M\in FMS(]0,1])$, we shall denote by M/x the fuzzy multiset over X defined by $\overline{M}(x)=M$ and $\overline{M}(y)=\bot$ for every $y\neq x$.

We can generalize the axiomatic notion of scalar and fuzzy cardinalities of crisp multisets to fuzzy multisets by imposing an additivity condition and to behave like a cardinality of crisp multisets on the fuzzy multisets of the form M/x. Let us fix from now on a crisp set X.

Definition 5 *A* scalar cardinality *on* $\mathcal{FFMS}(X)$ *is a mapping* $Sc : \mathcal{FFMS}(X) \to \mathbb{R}^+$ *that satisfies the following conditions:*

(i) $Sc(\overline{A} + \overline{B}) = Sc(\overline{A}) + Sc(\overline{B})$ *for every* $\overline{A}, \overline{B} \in \mathcal{FFMS}(X)$.

(ii) $Sc((1/1)/x) = 1$ *for every* $x \in X$.

A scalar cardinality Sc *on* $\mathcal{FFMS}(X)$ *is* homogeneous *when it satisfies the following extra property:*

(iii) $Sc(M/x) = Sc(M/y)$ *for every* $x, y \in X$ *and* $M \in FMS(]0,1])$.

The thesis in Remarks 1 and 2 still hold for scalar cardinalities on $\mathcal{FFMS}(X)$, because they are direct consequences of the additivity property. In particular, $Sc(\overline{\perp}) = 0$ for every scalar cardinality Sc on $\mathcal{FFMS}(X)$.

Next proposition provides a description of all scalar cardinalities on $\mathcal{FFMS}(X)$.

Proposition 22 *A mapping* $Sc : \mathcal{FFMS}(X) \to \mathbb{R}^+$ *is a scalar cardinality if and only if for every* $x \in X$ *there exists an scalar cardinality* $Sc_x : FMS(]0,1])$ *such that*

$$Sc(\overline{M}) = \sum_{x \in X} Sc_x(\overline{M}(x)).$$

Moreover, the family $(Sc_x)_{x \in X}$ *is uniquely determined by* Sc*, and* Sc *is homogeneous if and only if* $Sc_x = Sc_y$ *for every* $x, y \in X$.

Proof. Let Sc be a scalar cardinality on $\mathcal{FFMS}(X)$, and consider for every $x \in X$ the mapping

$$\begin{array}{rccl} Sc_x : & FMS(]0,1]) & \to & \mathbb{R}^+ \\ & M & \mapsto & Sc(M/x) \end{array}$$

Conditions (i) and (ii) in Definition 5 entail that each Sc_x satisfy conditions (i) and (ii) in Definition 1, and hence that each Sc_x is a scalar cardinality on $FMS(]0,1])$. Now, it is straightforward to check that, for every $\overline{M} \in \mathcal{FFMS}(X)$,

$$\overline{M} = \sum_{x \in X} \overline{M}(x)/x$$

(notice that if $x \notin Supp(\overline{M})$, then $\overline{M}(x)/x = \overline{\perp}$, the null fuzzy multiset over X, and hence in this sum all fuzzy multisets are $\overline{\perp}$ except a finite set of them). Thus, the additivity property of Sc implies that

$$Sc(\overline{M}) = \sum_{x \in X} Sc(\overline{M}(x)/x) = \sum_{x \in X} Sc_x(\overline{M}(x)).$$

And notice that if Sc is homogeneous, then $Sc_x = Sc_y$ for every $x, y \in X$.

Conversely, for every $x \in X$ let $Sc_x : FMS(]0,1]) \to \mathbb{R}^+$ be a scalar cardinality, and let $Sc : \mathcal{FFMS}(X) \to \mathbb{R}^+$ be the mapping defined by

$$Sc(\overline{M}) = \sum_{x \in X} Sc_x(\overline{M}(x))$$

for every $\overline{M} \in \mathcal{FFMS}(X)$. This mapping satisfies the defining conditions of scalar cardinalities on $\mathcal{FFMS}(]0,1])$:

(i) For every $\overline{A}, \overline{B} \in \mathcal{FFMS}(X)$,

$$\begin{aligned} Sc(\overline{A}+\overline{B}) &= \textstyle\sum_{x\in X} Sc_x((\overline{A}+\overline{B})(x)) \\ &= \textstyle\sum_{x\in X} Sc_x((\overline{A}(x)/x) + (\overline{B}(x)/x)) \\ &= \textstyle\sum_{x\in X} (Sc_x(\overline{A}(x)/x) + Sc_x(\overline{B}(x)/x)) \quad \text{(by Definition 1.(i))} \\ &= \textstyle\sum_{x\in X} Sc_x(\overline{A}(x)/x) + \sum_{x\in X} Sc_x(\overline{B}(x)/x) = Sc(\overline{A}) + Sc(\overline{B}) \end{aligned}$$

(ii) $Sc((1/1)/x) = Sc_x(1/1) = 1$ by Definition 1.(ii).

Now notice that

$$Sc(M/x) = \sum_{y \in X} Sc_y((M/x)(y)) = Sc_x(M) + \sum_{y \neq x} Sc_y(\perp) = Sc_x(M),$$

which, together with the "only if" implication proved above, implies that every Sc_x is uniquely determined by Sc. And in particular, if $Sc_x = Sc_y$ for every $x, y \in M$, then Sc is homogeneous. ∎

Corollary 23 *A mapping $Sc : \mathcal{FFMS}(X) \to \mathbb{R}^+$ is a homogeneous scalar cardinality on $\mathcal{FFMS}(X)$ if and only if there exists a scalar cardinality on $FMS(]0,1])$, which we still denote by Sc, such that*

$$Sc(\overline{M}) = Sc(\sum_{x \in X} \overline{M}(x)) = \sum_{x \in X} Sc(\overline{M}(x)).$$

Let us move now to fuzzy cardinalities.

Definition 6 *A* fuzzy cardinality *on $\mathcal{FFMS}(X)$ is a mapping $\mathcal{C} : \mathcal{FFMS}(X) \to \overline{\mathbb{N}}$ that satisfies the following conditions:*

(i) For every $\overline{A}, \overline{B} \in \mathcal{FFMS}(X)$, $\mathcal{C}(\overline{A}+\overline{B}) = \mathcal{C}(\overline{A}) \oplus \mathcal{C}(\overline{B})$.

(ii) For every $x \in X$, the mapping

$$\begin{array}{rrcl} \mathcal{C}(\ /x): & FMS(]0,1]) & \to & \overline{\mathbb{N}} \\ & M & \mapsto & \mathcal{C}(M/x) \end{array}$$

is a fuzzy cardinality on $FM(]0,1])$

A fuzzy cardinality is homogeneous *when it satisfies the following further condition:*

(iii) For every $x, y \in X$, $\mathcal{C}(\ /x) = \mathcal{C}(\ /y)$.

A simple argument, similar to the proof of Proposition 22, and which we leave to the reader, proves the following result.

Proposition 24 *A mapping $\mathcal{C} : \mathcal{FFMS}(X) \to \overline{\mathbb{N}}$ is a fuzzy cardinality if and only if for every $x \in X$ there exists an fuzzy cardinality $\mathcal{C}_x : FMS(]0,1]) \to \overline{\mathbb{N}}$ such that*

$$\mathcal{C}(\overline{M}) = \bigoplus_{x \in X} \mathcal{C}_x(\overline{M}(x)).$$

Moreover, the family $(\mathcal{C}_x)_{x \in X}$ is uniquely determined by $\mathcal{C}$, and $\mathcal{C}$ is homogeneous if and only if $\mathcal{C}_x = \mathcal{C}_y$ for every $x, y \in X$.

Corollary 25 *A mapping $\mathcal{C} : \mathcal{FFMS}(X) \to \overline{\mathbb{N}}$ is a homogeneous fuzzy cardinality on $\mathcal{FFMS}(X)$ if and only if there exists a fuzzy cardinality on $FMS(]0,1])$, which we still denote by $\mathcal{C}$, such that*

$$\mathcal{C}(\overline{M}) = \mathcal{C}(\sum_{x \in X} \overline{M}(x)) = \bigoplus_{x \in X} \mathcal{C}(\overline{M}(x)).$$

Thus, homogeneous scalar and fuzzy cardinalities understand fuzzy multisets as a sum of crisp multisets, one on every type $x \in X$, and "count" this sum. Arbitrary scalar and fuzzy cardinalities "count" each multiset on each $x \in X$, possibly using a different cardinality for every $x \in X$, and then add up these results.

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