Generalized Fuzzy Multisets and P Systems

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Abstract

By fuzzifying the number of occurrences of an element of a multiset, we obtain a new fuzzy structure; similarly, by fuzzifying the number of occurrences of an element of a hybrid set, we obtain another new fuzzy structure. The aim of the present work is twofold: to provide a concise definition of these new fuzzy structures and their properties and to apply them in natural computing. More specifically, general fuzzy P systems are introduced that are capable of computing any real number.

1 Introduction

Intuitively, a set is a collection of elements (e.g., numbers or symbols) that is completely determined by them.[1] The elements of a set are pairwise different. If we relax this restriction and allow repeated occurrences of any element, then we end up with a mathematical structure that is known as *multiset*[2] (see [2] for a historical account of the development of the multiset theory; also, see [3] for a recent account of the mathematical theory of multisets). Multisets are really useful structures and they have found numerous application in mathematics and computer science. For example, the prime factorization of an integer $n > 0$ is a multiset $\mathcal{N}$ whose elements are primes. Also, every monic polynomial $f(x)$ over the complex numbers corresponds in a natural way to the multiset $\mathcal{F}$ of its roots. In addition, multisets have been used in concurrency theory [4]. A rather interesting recent development in the theory of multisets is the discovery that the logic of multisets is the $\{\otimes, \multimap, \oplus, \mathbf{1}\}$-fragment of intuitionistic linear logic (see [5, 6] for details).

If we allow elements of a multiset to occur an integer number of times (and that includes a *negative* number of times), we end up with a structure that is called *hybrid set*. These structures have been introduced by Loeb [7]. Initially, one may wonder whether hybrid sets are of any use. However, Loeb has shown that they are indeed very useful structures (see [7, 8]). For example, one can use a hybrid set to describe the roots and the poles of a rational function. In particular, if $f(x)$ is a monic rational function, then $f(x)$ can be written in terms of its roots $a_1, a_2, \ldots, a_n$, and its poles $b_1, b_2, \ldots, b_m$, as follows:

$$f(x) = c\frac{(x-a_1)(x-a_2)\cdots(x-a_n)}{(x-b_1)(x-b_2)\cdots(x-b_m)}$$

[1]For the present discussion this vague definition is adequate, but it may lead to paradoxes like the "set of all sets" paradox, which is known in the literature as Russell's paradox. However, such paradoxes will not concern us here.

[2]The term "multiset" has been coined by N.G. de Bruijn [1].

We can now form a hybrid set where elements that occur a positive number of times correspond to roots of the function, while elements that occur a negative number of times corresponds to the poles of the function.

In his seminal paper [9], Yager introduced fuzzy multisets, that is fuzzy subsets where an element may occur more than one time (see [10] for an up-to-date presentation of the theory of fuzzy multisets, which, however, does not differ significantly from [11]). From a mathematical point of view, fuzzy multisets are just multisets of pairs, where the first part of each pair is an element of the universe set and the second part the degree to which the first part belongs to the fuzzy multiset. Practically, this means that fuzzy multisets are not fuzzy enough. Nevertheless, these structures are useful as they can be utilized to model everyday experience. For example, if we have a group of people and we want to classify them according to their height, then a natural choice would be a fuzzy multiset. However, if we have a group of people that have voted for one party between a number of different parties and at the same time we lack precise information on the total number of people that have voted a particular party, then fuzzy multisets are useless. Clearly, we need a set structure where the elements are repeated to a certain degree (e.g., 75 people voted party A with degree that is equal to 0.89). More generally, in the case where certain votes may have a negative impact on the final result, we may opt to model the situation with a hybrid set where the elements are repeated to a certain degree. I believe that cases like this are not uncommon, therefore it is really useful to formally define mathematical structures that can be used to rigorously model such cases.

Membrane computing is a computational paradigm that was inspired by the way cells live and function (see [12] for an overview of the field of membrane computing). Roughly speaking, a cell consists of a membrane that separates the cell from its environment. In addition, this membrane consists of compartments surrounded by membranes, which, in turn, may contain other compartments, and so on. At any moment, matter flows from one compartment to any neighboring one. In addition, the cell interacts with its environment in various ways (e.g., by dumping matter to its environment). Obviously, at any moment a number of processes occur in parallel (e.g., matter moves into a compartment, while energy is consumed in another compartment, etc.).

Generally speaking, a P system is a computational device, which is an abstraction of a cell. Thus, a P system consists of membranes that are populated with multisets of objects, which are usually materialized as strings of symbols. In addition, there are rules that are used to change the configuration of the system. A P system behaves more or less like a parser, which is clearly hard-wired to a particular grammar. Thus, a P system stops when no rule can be applied to the system. The result of the computation is always equal to the cardinality of the multiset that is contained in a designated compartment. Now, since rigid mathematical models employed in life sciences are not completely adequate for the interpretation of biological information, there have been various proposals to use fuzzy sets in the modelling of biological systems (e.g., see [13, 14]). Thus, it quite reasonable to attempt the use of the theory of fuzzy sets in P systems. Indeed, in [15] the author presents such a first attempt, which is generalized to a certain degree in this paper.

Structure of the paper In what follows, I will define L-multi-fuzzy sets and L-fuzzy hybrid sets. Next, I will define the basic operations between such structures (e.g., union, sum, etc.). Also, I will give the definition of certain standard fuzzy-theoretic operators. The use of these structures in the definition of fuzzy variants of P systems and their properties will be presented next. The paper ends with the customary concluding remarks.

2 On L-Multi Fuzzy Sets and L-Fuzzy Hybrid Sets

A multiset is a collection of possibly repeated elements. We may view a multiset as a collection of *tokens* of various *types* that make up a set. In particular, if a multiset contains n copies of a, we may say that the multiset contains n tokens of type a (see [5] for more details). Formally, a multiset F is a function $F : A \to \mathbb{N}_0$, where A is a set from which we draw the elements of F and $\mathbb{N}_0 = \{0, 1, 2, 3, \ldots\}$ (i.e., the set of non-negative integers). Similarly, a hybrid set G is a function $G : A \to \mathbb{Z}$, where $\mathbb{Z}$ is the set of integers. In addition, a fuzzy multiset H is a function $A \times \mathrm{I} \to \mathbb{N}$, where $\mathrm{I} = [0, 1]$. Assume that $\mathcal{M}(\mathrm{I})$ denotes the collection of all finite multisets that can be created from the set I, then a fuzzy multiset H is characterized by the "function" $C_H : A \to \mathcal{M}(\mathrm{I})$. In other words, given an element $a \in A$,

$$C_H(a) = \{i_1^{l_1}, i_2^{l_2}, \ldots, i_n^{l_n}\}, \quad i_1^{l_1}, i_2^{l_2}, \ldots, i_n^{l_n} \in \mathrm{I}.$$

Clearly, one can write H as follows:

$$H = \{(a, i_1^{l_1}), (a, i_2^{l_2}), \ldots, (a, i_n^{l_n}), \ldots\}.$$

Alternatively, one can write the multiset above in the following more compact form:

$$H = \{(i_1^{l_1}, i_2^{l_2}, \ldots, i_n^{l_n})/a, \ldots\}.$$

Nevertheless this does not really changes the essence of the structure, which in my own opinion is not fuzzy enough. In order to have a fuzzier structure, one way is to fuzzify the number of occurrences of repeated elements. One benefit of such an approach is that it is in the spirit of the theory of fuzzy sets, where one assigns a membership grade to each element of a crisp set. Clearly, this can be achieved by generalizing the membership function. Naturally, a function $H : A \to \mathrm{I} \times \mathbb{N}$ is a generalization of the notion of multisets and one that is in the spirit of the theory of fuzzy sets. Since, the term "fuzzy multiset" has been used for the structures I described above, I have decided to call these new structures "multi-fuzzy sets" [15]. However, it is my belief that these structures are not general enough as they cannot describe all possible cases. For example, one problem would be the description of a situation where certain elements have a rather negative impact on it. Obviously, if we can assign a negative occurrence to these elements, then we can clearly describe this case. In addition, if we associate with each number of occurrences n of an element a a "manifoldness" degree i, according to which we believe that a really occurs n times, then we obtain a truly fuzzy structure. In the most general case, we want the values of the "manifoldness" degree i to be drawn from a frame L. This way we get L-fuzzy structures, which is not something new—from fuzzy sets we "easily" get L-fuzzy sets [16]. Clearly, one can further fuzzify this structure, but I have no intention to pursue any further the fuzzification process, at least in the present work. Let us now proceed with the formal definition of L-fuzzy hybrid sets:

Definition 2.1 An L-fuzzy hybrid sets $\mathcal{A}$ is a function $\mathcal{A} : X \to L \times \mathbb{Z}$, where L is a frame. In particular, the equality $\mathcal{A}(x) = (\ell, n)$ means that there exists n copies of x to a degree that is equal to ℓ.

If we substitute $\mathbb{Z}$ with $\mathbb{N}_0$ in the previous definition, then the resulting structures will be called *L-multi-fuzzy sets.*

Assuming that $\mathcal{A}$ is L-fuzzy hybrid set, then one can define the following two functions: the *multiplicity* function $\mathcal{A}_m : X \to \mathbb{Z}$ and the *membership* function $\mathcal{A}_\mu : X \to L$. Clearly, if $\mathcal{A}(x) =$

(ℓ, n), then $\mathcal{A}_m(x) = n$ and $\mathcal{A}_\mu(x) = \ell$. Note that it is equally easy to define the corresponding functions for a L-multi-fuzzy set.

Now that I have defined these new structures, it is time to prove my claim that fuzzy multisets are really a special case of these structures:

Corollary 2.1 *Fuzzy multisets are a special case of fuzzy hybrid sets.*

Proof. Assuming that $A : X \times \mathrm{I} \to \mathbb{N}_0$ is a fuzzy multiset, then the fuzzy hybrid set $\mathcal{A} : X \times \mathrm{I} \to \mathbb{1} \times \mathbb{Z}$, where $\mathbb{1}$ is the *inconsistent* frame consisting of only one element, is a faithful representation of A. In particular, if $A(x, i) = n$, then $\mathcal{A}(x, i) = (\top, n)$. □

I believe this is a good point to briefly express my prejudices and my intentions regarding the present work. Clearly, it is not my intention to develop an axiomatic set theory of L-fuzzy hybrid sets and L-multi-fuzzy sets, in the sense of the Zermelo-Frænkel set theory, but rather a "naïve" set theory in the sense that I will not present a precise axiomatization. Therefore, I plan to introduce only the basic set-theoretic operations and the basic properties of these sets. To begin with, let me now define the cardinality of a L-fuzzy hybrid set:

Definition 2.2 Assume that $\mathcal{A}$ is a L-fuzzy hybrid set that draws elements from a universe X, then its cardinality is defined as follows:

$$\operatorname{card} \mathcal{A} = \sum_{x \in X} \mathcal{A}_\mu(x) \otimes \mathcal{A}_m(x),$$

where $\ell \otimes n$ is a binary operator that maps $\ell \in L$ and $n \in \mathbb{Z}$ to some real number.

Note that in the case that X is actually a fuzzy multiset, the definition above does not produce the "expected" results. However, one should not forget that fuzzy multisets are actually crisp multisets.

The cardinality of a set is equal to the number of elements the set contains. Clearly, the previous definition is not in spirit with this assumption. However, hybrid sets may contain elements that occur a negative number of times. Thus, one may think that we should take this fact under consideration when computing the cardinality of a hybrid set and, more generally, the cardinality of a L-fuzzy hybrid set. So, it makes sense to introduce the notion of a *weak* cardinality defined as follows:

Definition 2.3 Assume that $\mathcal{A}$ is a L-fuzzy hybrid set that draws elements from a universe X, then its *w*eak cardinality is defined as follows:

$$\operatorname{card} \mathcal{A} = \sum_{x \in X} \mathcal{A}_\mu(x) \otimes |\mathcal{A}_m(x)|,$$

where $|\mathcal{A}_m(x)|$ denotes the absolute value of $\mathcal{A}_m(x)$.

For reasons of completeness I give below the definition of the cardinality of a L-multi-fuzzy set:

Definition 2.4 Assume that $\mathcal{A}$ is a L-multi-fuzzy set that draws elements from a universe X, then its cardinality is defined as follows:

$$\operatorname{card} \mathcal{A} = \sum_{x \in X} \mathcal{A}_\mu(x) \otimes \mathcal{A}_m(x),$$

where $\ell \otimes n$ is a binary operator that maps $\ell \in L$ and $n \in \mathbb{N}_0$ to some positive real number (since $n \geq 0$).

In order to complete the presentation of the basic properties of fuzzy hybrid sets, it is necessary to define the notion of subsethood. Before, going on with this definition, I will introduce the (new) partial order $\ll$ over $\mathbb{Z}$. In particular, if $n, m \in \mathbb{Z}$, then

$$\begin{aligned} n \ll m \quad \equiv \quad & (n = 0) \vee \\ & \Big((n > 0) \wedge (m > 0) \wedge (n \leq m)\Big) \vee \\ & \Big((n < 0) \wedge (m > 0)\Big) \vee \\ & \Big((n < 0) \wedge (m < 0) \wedge (|n| \leq |m|)\Big). \end{aligned}$$

Note that here $\wedge$ and $\vee$ denote the classical logical conjunction and disjunction operators, respectively. In addition, $|n|$ denotes the absolute value of n and $\leq$, $<$ denote the usual integer ordering relations.

Proposition 2.1 *The relation $\ll$ is a partial order.*

Proof. I have to prove that the relation $\ll$ is reflexive, antisymmetric and transitive:

Reflexivity Assume that $a \in \mathbb{Z}$, then if $a = 0$, $a \ll a$ from the first part of the disjunction. If $a < 0$, then $a \ll a$ from the fourth part of the disjunction and if $a > 0$, then $a \ll a$ from the second part of the disjunction.

Antisymmetry Assume that $a, b \in \mathbb{Z}$, $a \ll b$, and $b \ll a$, then if $a = 0$ this implies that $b = 0$ and so $a = b$. If $a < 0$, then it follows that $b < 0$, $|a| \leq |b|$, and $|b| \leq |a|$, which implies that $a = b$. Similarly, if $a > 0$, then it follows that $b > 0$, $a \leq b$, and $b \leq a$, which implies that $a = b$.

Transitivity Assume that $a, b, c \in \mathbb{Z}$, $a \ll b$, and $b \ll c$, then if $a = 0$, then clearly $a \ll c$. If $a < 0$ and $b < 0$, then either $c < 0$ or $c > 0$, but since $|b| \leq |c|$, this implies that $a \ll c$. If $a > 0$ and $b > 0$, then $c > 0$ and since $b \leq c$ this implies that $a \ll c$. If $a < 0$ and $b > 0$, then since $b \ll c$, this implies that $c > 0$, which means that $a \ll c$.

□

Note that $a \gg b$ is an alternative form of $b \ll a$, which will be used in the rest of this paper. Let us now proceed with the definition of the notion of subsethood for L-fuzzy hybrid sets:

Definition 2.5 Assume that $\mathcal{A}, \mathcal{B} : X \to L \times \mathbb{Z}$ are two L-fuzzy hybrid sets, then $\mathcal{A} \subseteq \mathcal{B}$ if and only if $\mathcal{A}_\mu(x) \sqsubseteq \mathcal{B}_\mu(x)$ and $\mathcal{A}_m(x) \ll \mathcal{B}_m(x)$ for all $x \in X$.

Remark that for all $\ell_1, \ell_2 \in L$, $\ell_1 \sqsubseteq \ell_2$ if ℓ_1 is "less than or equal" to ℓ_2 in the sense of the partial order defined over L. The definition of subsethood for L-multi-fuzzy sets is more straightforward:

Definition 2.6 Assume that $\mathcal{A}, \mathcal{B} : X \to L \times \mathbb{N}_0$ are two L-multi-fuzzy sets, then $\mathcal{A} \subseteq \mathcal{B}$ if and only if $\mathcal{A}_\mu(x) \sqsubseteq \mathcal{B}_\mu(x)$ and $\mathcal{A}_m(x) \leq \mathcal{B}_m(x)$ for all $x \in X$.

3 Basic Set Operations

The basic operations between sets are their union and their intersection. A third operation, viz. set sum, is meaningful only for multisets. Also, since both L-multi-fuzzy sets and L fuzzy hybrid sets are actually generalizations of fuzzy sets, one should be able to define the α-cuts of such sets. I will start by defining the basic set operations between L-multi-fuzzy sets.

3.1 Set Operations of L-Multi-Fuzzy Sets

Let me first present the definitions of union and intersection of L-multi-fuzzy sets:

Definition 3.1 Assuming that $\mathcal{A}, \mathcal{B} : X \rightarrow L \times \mathbb{N}_0$ are two L-multi-fuzzy sets, then their union, denoted $\mathcal{A} \cup \mathcal{B}$, is defined as follows:

$$(\mathcal{A} \cup \mathcal{B})(x) = \Big(\mathcal{A}_\mu(x) \sqcup \mathcal{B}_\mu(x), \max\Big\{\mathcal{A}_m(x), \mathcal{B}_m(x)\Big\}\Big),$$

where $a \sqcup b$ is the join of $a, b \in L$.

Definition 3.2 Assuming that $\mathcal{A}, \mathcal{B} : X \rightarrow L \times \mathbb{N}_0$ are two L-multi-fuzzy sets, then their intersection, denoted $\mathcal{A} \cap \mathcal{B}$, is defined as follows:

$$(\mathcal{A} \cap \mathcal{B})(x) = \Big(\mathcal{A}_\mu(x) \sqcap \mathcal{B}_\mu(x), \min\Big\{\mathcal{A}_m(x), \mathcal{B}_m(x)\Big\}\Big),$$

where $a \sqcap b$ is the meet of $a, b \in L$.

I will now define the sum of two L-multi-fuzzy sets:

Definition 3.3 Suppose that $\mathcal{A}, \mathcal{B} : X \rightarrow L \times \mathbb{N}_0$ are two L-multi-fuzzy sets, their sum, denoted $\mathcal{A} \uplus \mathcal{B}$, is defined as follows:

$$(\mathcal{A} \uplus \mathcal{B})(x) = \Big(\mathcal{A}_\mu(x) \sqcup \mathcal{B}_\mu(x), \mathcal{A}_m(x) + \mathcal{B}_m(x)\Big).$$

Although it is crystal clear, it is necessary to say that $\sqcup$ and $\sqcap$ are operators that are part of the definition of the frame L. And as such they have a number of properties (e.g., they are idempotent, etc., see [17, p. 15] for details) that, naturally, affect the properties of the operations defined so far. Indeed, these operations have the following properties:

Theorem 3.1 *For any three L-multi-fuzzy sets $\mathcal{A}, \mathcal{B}, \mathcal{C} : X \rightarrow L \times \mathbb{N}_0$ the following equalities hold:*

i) Commutativity:

$$\begin{aligned} \mathcal{A} \cup \mathcal{B} &= \mathcal{B} \cup \mathcal{A} \\ \mathcal{A} \cap \mathcal{B} &= \mathcal{B} \cap \mathcal{A} \\ \mathcal{A} \uplus \mathcal{B} &= \mathcal{B} \uplus \mathcal{B}; \end{aligned}$$

ii) Associativity:

$$\begin{aligned} \mathcal{A} \cup (\mathcal{B} \cup \mathcal{C}) &= (\mathcal{A} \cup \mathcal{B}) \cup \mathcal{C} \\ \mathcal{A} \cap (\mathcal{B} \cap \mathcal{C}) &= (\mathcal{A} \cap \mathcal{B}) \cap \mathcal{C} \\ \mathcal{A} \uplus (\mathcal{B} \uplus \mathcal{C}) &= (\mathcal{A} \uplus \mathcal{B}) \uplus \mathcal{C}; \end{aligned}$$

iii) Idempotency:

$$\begin{aligned} \mathcal{A} \cup \mathcal{A} &= \mathcal{A} \\ \mathcal{A} \cap \mathcal{A} &= \mathcal{A}; \end{aligned}$$

iv) Distributivity:

$$\begin{aligned}\mathcal{A} \cap (\mathcal{B} \cup \mathcal{C}) &= (\mathcal{A} \cap \mathcal{B}) \cup (\mathcal{A} \cap \mathcal{C}) \\ \mathcal{A} \cup (\mathcal{B} \cap \mathcal{C}) &= (\mathcal{A} \cup \mathcal{B}) \cap (\mathcal{A} \cup \mathcal{C});\end{aligned}$$

v) Distributivity of sum:

$$\begin{aligned}\mathcal{A} \uplus (\mathcal{B} \cup \mathcal{C}) &= (\mathcal{A} \uplus \mathcal{B}) \cup (\mathcal{A} \uplus \mathcal{C}) \\ \mathcal{A} \uplus (\mathcal{B} \cap \mathcal{C}) &= (\mathcal{A} \uplus \mathcal{B}) \cap (\mathcal{A} \uplus \mathcal{C});\end{aligned}$$

Proof.

i) Although this is easy, I will prove all cases:

$$\begin{aligned}(\mathcal{A} \cup \mathcal{B})(z) &= \Big(\mathcal{A}_\mu(z) \sqcup \mathcal{B}_\mu(z), \max\Big\{\mathcal{A}_m(z), \mathcal{B}_m(z)\Big\}\Big) \\ &= \Big(\mathcal{B}_\mu(z) \sqcup \mathcal{A}_\mu(z), \max\Big\{\mathcal{B}_m(z), \mathcal{A}_m(z)\Big\}\Big) \\ &= (\mathcal{B} \cup \mathcal{A})(z) \\ (\mathcal{A} \cap \mathcal{B})(z) &= \Big(\mathcal{A}_\mu(z) \sqcap \mathcal{B}_\mu(z), \min\Big\{\mathcal{A}_m(z), \mathcal{B}_m(z)\Big\}\Big) \\ &= \Big(\mathcal{B}_\mu(z) \sqcap \mathcal{A}_\mu(z), \min\Big\{\mathcal{B}_m(z), \mathcal{A}_m(z)\Big\}\Big) \\ &= (\mathcal{B} \cap \mathcal{A})(z) \\ (\mathcal{A} \uplus \mathcal{B})(z) &= \Big(\mathcal{A}_\mu(z) \sqcup \mathcal{B}_\mu(z), \mathcal{A}_m(z) + \mathcal{B}_m(z)\Big) \\ &= \Big(\mathcal{B}_\mu(z) \sqcup \mathcal{A}_\mu(z), \mathcal{B}_m(z) + \mathcal{A}_m(z)\Big) \\ &= (\mathcal{B} \uplus \mathcal{A})(z)\end{aligned}$$

ii) I will prove only the first case as the others can be proved similarly:

$$\begin{aligned}(\mathcal{A} \cup (\mathcal{B} \cup \mathcal{C}))(z) &= \Big(\mathcal{A}_\mu(z) \sqcup \Big(\mathcal{B}_\mu(z) \sqcup \mathcal{C}_\mu(z)\Big), \max\Big\{\mathcal{A}_m(z), \max\Big\{\mathcal{B}_m(z), \mathcal{C}_m\Big\}\Big\}\Big) \\ &= \Big(\Big(\mathcal{A}_\mu(z) \sqcup \mathcal{B}_\mu(z)\Big) \sqcup \mathcal{C}_\mu(z), \max\Big\{\max\Big\{\mathcal{A}_m(z), \mathcal{B}_m(z)\Big\}, \mathcal{C}_m\Big\}\Big) \\ &= ((\mathcal{A} \cup \mathcal{B}) \cup \mathcal{C})(z)\end{aligned}$$

iii) As in the previous case, I will prove only the first case as the other can be proved similarly:

$$\begin{aligned}(\mathcal{A} \cup \mathcal{A})(z) &= \Big(\mathcal{A}_\mu(z) \sqcup \mathcal{A}_\mu(z), \max\{\mathcal{A}_m(z), \mathcal{A}_m(z)\}\Big) \\ &= \Big(\mathcal{A}_\mu(z), \mathcal{A}_m(z)\Big) \\ &= \mathcal{A}(z)\end{aligned}$$

iv) The proof of this case follows from the fact that the following equalities are true for the any three elements of a frame:

$$\begin{aligned} x \sqcap (y \sqcup z) &= (x \sqcap y) \sqcup (x \sqcap y) \\ x \sqcup (y \sqcap z) &= (x \sqcup y) \sqcap (x \sqcup y) \end{aligned}$$

v) As with the previous case the proof for this case follows from the fact that for any $x, y, z \in \mathbb{N}_0$ the following equalities hold:

$$\begin{aligned} x + \max\{y, z\} &= \max\{x + y, x + z\} \\ x + \min\{y, z\} &= \min\{x + y, x + z\} \end{aligned}$$

□

The α-cut of a fuzzy subset is just a crisp set. Similarly, the α-cut of a L-multi-fuzzy set has to be a multiset. Indeed, if we assume that $[x]_n$ denotes a multiset that consists of only n copies of x, the following definition is in spirit with the general theory of fuzzy sets:

Definition 3.4 Suppose that $\mathcal{A}$ is L-multi-fuzzy set with universe the set X, also assume that $\alpha \in L$, then the α-cut of $\mathcal{A}$, denoted by ${}^{\alpha}\!\mathcal{A}$, is the multiset

$${}^{\alpha}\!\mathcal{A} = \bigcup_{\substack{x \in X \\ \alpha \sqsubseteq \mathcal{A}_\mu(x)}} [x]_{\mathcal{A}_m(x)}.$$

Not so surprisingly, the properties of the α-cut of L-multi-fuzzy sets are similar to those of plain fuzzy sets. These properties are summarized below:

Theorem 3.2 *Assume that $\mathcal{A}$ and $\mathcal{B}$ are two L-multi-fuzzy sets with universe the set X, then the following properties hold:*

i) if $\alpha \sqsubseteq \beta$, then ${}^{\alpha}\!\mathcal{A} \sqsupseteq {}^{\beta}\!\mathcal{A}$

ii) ${}^{\alpha}(\mathcal{A} \cap \mathcal{B}) = {}^{\alpha}\!\mathcal{A} \cap {}^{\alpha}\mathcal{B}$, ${}^{\alpha}(\mathcal{A} \cup \mathcal{B}) = {}^{\alpha}\!\mathcal{A} \cup {}^{\alpha}\mathcal{B}$, and ${}^{\alpha}(\mathcal{A} \uplus \mathcal{B}) = {}^{\alpha}\!\mathcal{A} \uplus {}^{\alpha}\mathcal{B}$.

Proof.

i) Let $x \in X$ and assume that $\alpha \sqsubseteq \beta$. If $\mathcal{A}_\mu(x) \not\sqsubseteq \beta$, then ${}^{\alpha}\!\mathcal{A}(x) = {}^{\beta}\!\mathcal{A}(x)$. If $\alpha \sqsubseteq \mathcal{A}_\mu(x) \sqsubseteq \beta$, then ${}^{\alpha}\!\mathcal{A}(x) \geq {}^{\beta}\!\mathcal{A}(x)$. If $\alpha \not\sqsubseteq \mathcal{A}_\mu(x)$, then ${}^{\alpha}\!\mathcal{A}(x) = {}^{\beta}\!\mathcal{A}(x) = 0$. Thus, for all possible cases ${}^{\alpha}\!\mathcal{A}(x) \geq {}^{\beta}\mathcal{A}(x)$, which means that ${}^{\alpha}\!\mathcal{A} \sqsupseteq {}^{\beta}\!\mathcal{A}$.

ii) Assume that ${}^{\alpha}(\mathcal{A} \cap \mathcal{B})(x) > 0$, then $(\mathcal{A} \cap \mathcal{B})_\mu(x) \sqsupseteq \alpha$ and hence $\mathcal{A}_\mu(x) \sqcap \mathcal{B}_\mu(x) \sqsupseteq \alpha$. This means that $\mathcal{A}_\mu(x) \sqsupseteq \alpha$ and $\mathcal{B}_\mu(x) \sqsupseteq \alpha$, which implies that $({}^{\alpha}\!\mathcal{A} \cap {}^{\alpha}\mathcal{B})(x) > 0$. In other words,

$$\min\Big\{\mathcal{A}_m(x), \mathcal{B}_m(x)\Big\} \leq \min\Big\{{}^{\alpha}\!\mathcal{A}(x), {}^{\alpha}\mathcal{B}(x)\Big\}. \qquad (\alpha)$$

Conversely, if we assume that $({}^{\alpha}\!\mathcal{A} \cap {}^{\alpha}\mathcal{B})(x) > 0$, then ${}^{\alpha}\!\mathcal{A}(x) > 0$ and ${}^{\alpha}\mathcal{B}(x) > 0$; this means that $\mathcal{A}_\mu(x) \sqsupseteq \alpha$ and $\mathcal{B}_\mu(x) \sqsupseteq \alpha$, hence, $\mathcal{A}_\mu(x) \sqcap \mathcal{B}_\mu(x) \sqsupseteq \alpha$. This means that $(\mathcal{A} \cap \mathcal{B})_\mu(x) \sqsupseteq \alpha$, that is ${}^{\alpha}(\mathcal{A} \cap \mathcal{B})(x) > 0$. In other words,

$$\min\Big\{\mathcal{A}_m(x), \mathcal{B}_m(x)\Big\} \geq \min\Big\{{}^{\alpha}\!\mathcal{A}(x), {}^{\alpha}\mathcal{B}(x)\Big\}. \qquad (\beta)$$

From (α) and (β) we conclude that

$$\min\Big\{\mathcal{A}_m(x), \mathcal{B}_m(x)\Big\} = \min\Big\{{}^{\alpha}\mathcal{A}(x), {}^{\alpha}\mathcal{B}(x)\Big\},$$

which proves the first equality. Similarly, we can prove the other equalities.

□

3.2 Set Operations of L-Fuzzy Hybrid Sets

Loeb has shown that the set of all subsets of a given hybrid set with the subsethood relation do not form a lattice. This means that if f and g are two hybrid sets, then if they have lower bounds, they do not necessarily have a greatest lower bound. Similarly, if f and g have upper bounds, then they do not necessarily have a lowest upper bound. Practically, this means that given two hybrid sets f and g, one cannot define their union and their intersection. Fortunately, the sum of hybrid sets is a well-defined operation. Thus, we can easily extend this definition as follows:

Definition 3.5 Assume that $\mathcal{A}, \mathcal{B} : X \to L \times \mathbb{Z}$ are two L-fuzzy hybrid sets, their sum, denoted $\mathcal{A} \uplus \mathcal{B}$, is defined as follows:

$$(\mathcal{A} \uplus \mathcal{B})(x) = \Big(\mathcal{A}_\mu(x) \sqcup \mathcal{B}_\mu(x), \mathcal{A}_m(x) + \mathcal{B}_m(x)\Big).$$

Assume that $\{f_i\}$ denotes a finite collection of hybrid sets with a common universe X, where each of these sets contains repeated occurrence of only one element $x_i \in X$. In addition, let us insist that no two f_i and f_j will have common elements. Also, let us denote with $\uplus_i f_i$ the unique hybrid set that is the sum of all f_i. With these preliminary definitions, the road for the following definition has been paved:

Definition 3.6 Suppose that $\mathcal{A}$ is L-fuzzy hybrid set with universe the set X and that $\alpha \in L$, then the α-cut of $\mathcal{A}$, denoted by ${}^{\alpha}\mathcal{A}$, is the hybrid set $\uplus_i f_i$, where $f_i(x_i) = \mathcal{A}_m(x)$ iff $\alpha \sqsubseteq \mathcal{A}_\mu(x)$, for all $x_i \in X$.

The α-cut of L-fuzzy hybrid sets has the following properties:

Theorem 3.3 *Assume that $\mathcal{A}$ and $\mathcal{B}$ are two L-fuzzy hybrid sets with universe the set X, then the following properties hold:*

i) if $\alpha \sqsubseteq \beta$, then ${}^{\alpha}\mathcal{A} \sqsupseteq {}^{\beta}\mathcal{A}$

ii) ${}^{\alpha}(\mathcal{A} \uplus \mathcal{B}) = {}^{\alpha}\mathcal{A} \uplus {}^{\alpha}\mathcal{B}$.

Proof. I will prove only the second statement. Assume that ${}^{\alpha}(\mathcal{A} \uplus \mathcal{B}) \gg 0$, then $(\mathcal{A} \uplus \mathcal{B})_\mu(x) \sqsupseteq \alpha$ and, hence, $\mathcal{A}_m(x) \sqcup \mathcal{B}_\mu(x) \sqsupseteq \alpha$. This means that either $\mathcal{A}_\mu(x) \sqsupseteq \alpha$ or $\mathcal{B}_\mu(x) \sqsupseteq \alpha$ or even both $\mathcal{A}_\mu(x) \sqsupseteq \alpha$ and $\mathcal{B}_\mu(x) \sqsupseteq \alpha$. Clearly, this implies that $({}^{\alpha}\mathcal{A} \uplus {}^{\alpha}\mathcal{B})(x) \gg 0$. In other words,

$$\mathcal{A}_m + \mathcal{B}_m(x) \ll {}^{\alpha}\mathcal{A}(x) + {}^{\alpha}\mathcal{B}(x) \qquad (\alpha')$$

Conversely, if we assume that $({}^{\alpha}\mathcal{A} \uplus {}^{\alpha}\mathcal{B})(x) \gg 0$, then ${}^{\alpha}\mathcal{A}(x) \gg 0$ and ${}^{\alpha}\mathcal{B}(x) \gg 0$; this means that $\mathcal{A}_\mu(x) \sqsupseteq \alpha$ and $\mathcal{B}_\mu(x) \sqsupseteq \alpha$ and hence $\mathcal{A}_\mu(x) \sqcup \mathcal{B}_\mu(x) \sqsupseteq \alpha$, which implies that $(\mathcal{A} \uplus \mathcal{B})_\mu(x) \sqsupseteq \alpha$, that is ${}^{\alpha}(\mathcal{A} \uplus \mathcal{B})(x) \gg 0$. In other words,

$$\mathcal{A}_m + \mathcal{B}_m(x) \gg {}^{\alpha}\mathcal{A}(x) + {}^{\alpha}\mathcal{B}(x) \qquad (\beta')$$

From the antisymmetry of $\ll$, and (α'), (β') it follows that

$$\mathcal{A}_m + \mathcal{B}_m(x) = {}^{\alpha}\!\mathcal{A}(x) + {}^{\alpha}\mathcal{B}(x)$$

That is, ${}^{\alpha}(\mathcal{A} \uplus \mathcal{B}) = {}^{\alpha}\!\mathcal{A} \uplus {}^{\alpha}\mathcal{B}$. □

4 General Fuzzy P Systems

In [15] the author has proposed fuzzified versions of P systems. The basic idea behind this particular attempt to fuzzify P systems is the substitution of one or of all ingredients of a P system with their fuzzy counterparts. From a purely computational point of view, it turns out that only P systems that process multi-fuzzy sets are interesting. The reason being the fact that these systems are capable of computing (positive) real numbers. However, the author in [15] did not address the question which real numbers can be computed by such systems? I have to emphasize that if one replaces the multi-fuzzy sets employed in the author's previous work with L-multi-fuzzy sets, the computational power of the resulting P systems will not be any "greater," however, these systems may be quite useful for modelling of living organisms. But, things can get really interesting if we consider P systems with L-fuzzy hybrid sets, in general. Let us begin with the definition of these systems:

Definition 4.1 A general fuzzy P system is a construction

$$\Pi_{\mathrm{FD}} = (O, \mu, w^{(1)}, \ldots, w^{(m)}, R_1, \ldots, R_m, i_0)$$

where:

i) O is an alphabet (i.e., a set of distinct entities) whose elements are called *objects*;

ii) μ is the membrane structure of degree $m \geq 1$; membranes are injectivelly labeled with succeeding natural numbers starting with one;

iii) $w^{(i)} : O \to L \times \mathbb{Z}$, $1 \leq i \leq m$, are L-fuzzy hybrid sets over O that are associated with each region i;

iv) R_i, $1 \leq i \leq m$, are finite sets of multiset rewriting rules (called *evolution rules*) over O. An evolution rule is of the form $u \to v$, $u \in O^*$ and $v \in O^*_{\mathrm{TAR}}$, where $O_{\mathrm{TAR}} = O \times \mathrm{TAR}$, $\mathrm{TAR} = \{\mathrm{here}, \mathrm{out}\} \cup \{\mathrm{in}_j | 1 \leq j \leq m\}$. The effect of each rule is the removal of the elements of the left-hand side of each rule from the "current" compartment and the introduction of the elements of right-hand side to the designated compartments;

v) $i_0 \in \{1, 2, \ldots, m\}$ is the label of an elementary membrane (i.e., a membrane that does not contain any other membrane), called the *output* membrane.

The really interesting thing with the systems described in [15] is that I haven't managed to find any limits on what can be actually computed. Remember, that a real number $x \in \mathbb{R}$ is called *computable* iff there is a computable sequence $(r_n)_{n \in \mathbb{N}}$ of rational numbers which converges to x effectively, that is for all $n \in \mathbb{N}$, $|x - r_n| < 2^{-n}$ (see [18, 19] for details). In other words, this means that not all real numbers are *computable*. However, one should not forget that the definition of *computability* is hard-wired to the computational capabilities of the Universal Turing Machine and the so called Church-Turing thesis, which dictates what can be and what cannot be computed.

Now, the crucial question is whether there are any limits that prohibit the computation of certain numbers with P systems?

First of all, I do believe that it is better to work with a very simple system that has only two compartments, the second being the output compartment. Assume we want to check whether it is possible to compute $x \in \mathbb{R}$. In order to accomplish this task, we need to populate the first compartment with a L-fuzzy hybrid set that has cardinality equal to x (for simplicity we may assume that $L = [0, 1]$). Now, all we need is a set of rules that will transfer the L-fuzzy hybrid set to the output compartment and after that to stop all activities. Clearly, the new problem is the construction of a suitable I-fuzzy hybrid set that has cardinality equal to a given real number x. This, in turn, can be solved as follows: Assume that $x \in (-1, 1)$, then if $x < 0$, the fuzzy hybrid set will contain only one element that occurs -1 times with degree equal to $|x|$. Similarly, one can construct a fuzzy hybrid set when $x > 0$. If $x \notin (-1, 1)$, then we can split the number into two numbers y and z that are equal to the integer part of x and the decimal part of x, respectively. And of course, it is now easy to construct a hybrid set that has the desired cardinality. Obviously, this means that the following statement is true:

Theorem 4.1 *General fuzzy P systems can compute any real number.*

In other words, this means that L-fuzzy hybrid P systems are *hypercomputers* (i.e., computational devices that can compute more numbers than the Universal Turing machine does, see [20] for an overview of the field of hypercomputation). Clearly, skeptics will argue that general fuzzy P systems are not real, but rather a mathematical curiosity. My response to this argument is that first of all conventional P systems can be easily simulated in a ditrisbuted environment (e.g., see [21] for a description of the implementation of such a simulation) and, thus, it is not difficult to implement a fazzified P system to the extend we can fully implement a Turing machine. Thus, these systems are as real as any other common conceptual computational device.

5 Conclusions

In this paper I have introduced L-multi-fuzzy sets and L-fuzzy hybrid sets as well as their basic operations. In addition, general fuzzy P systems have been introduced and it was shown that they can be used to compute any real number. I do not believe that this is something really new—it is just another indication that the current theory of computation is simply inadequate to describe all computational phenomena. After all, this has been elegantly demonstrated by Stein in his thought provoking paper [22]. In addition, I believe that need a paradigm shift in computer science so to encompass new "phenomena" and practices.

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